B Dynamic inflow model

B.1 Model of Øye

Introduction A change in the rotor loading will result in a change in the wake configuration and the induced velocities. For a change between two loading configurations, it will take time for the wake to go from one equilibrium state to another. This phenomenon is referred to as the *dynamic inflow* or *dynamic wake*. Using a vorticity formulation, the change of loading will imply a change of vorticity emitted into the wake. The new value of the vorticity propagates progressively downstream replacing the old values and hence the induced velocity from this vorticity changes progressively. The time scales involved in the dynamic wake are thus related to the convection velocity of the vorticity in the wake. Due to the difference in convection velocity in the wake it is expected that the time delay towards the tip is shorter than towards the root [4].

ECN differential model Snel and Schepers suggested the following differential equation for the axial induction factor based on results from the cylindrical vortex wake model [4]:

$$\frac{4R}{U_0} f_a(r/R) \frac{\mathrm{d}a}{\mathrm{d}t} + 4a(1-a) = C_t \tag{44}$$

where f_a is an integral function responsible for the time delay in induction. It is seen that the STT relation is obtained for a steady case. A similar equation is suggested by the authors for the tangential induction factor.

Dynamic model of Øye The dynamic model of Øye is presented in the review of Snel and Schepers [4] and the book of Hansen [5]. The model is written using two first order differential equations:

$$\boldsymbol{W}_{\mathrm{int}} + \tau_1 \frac{\mathrm{d}\boldsymbol{W}_{\mathrm{int}}}{\mathrm{d}t} = \boldsymbol{W}_{\mathrm{qs}} + k\tau_1 \frac{\mathrm{d}\boldsymbol{W}_{\mathrm{qs}}}{\mathrm{d}t} \tag{45}$$

$$\boldsymbol{W} + \tau_2 \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t} = \boldsymbol{W}_{\mathrm{int}} \tag{46}$$

where $\boldsymbol{W}$ is the actual induction at the rotor (at a given blade position and radial position), $\boldsymbol{W}_{\mathrm{qs}}$ is the quasi-steady induction and $\boldsymbol{W}_{\mathrm{int}}$ is an intermediate value coupling the quasi-steady and the actual inductions. The constant k is usually chosen as $k = 0.6$. The steady solution of the systems leads to $\boldsymbol{W} = \boldsymbol{W}_{\mathrm{qs}}$. within an unsteady BEM step, once the values of a_{qs} and a'_{qs} are computed, the quasi-steady induction vector is determined as

$$\boldsymbol{W}_{\mathrm{qs}} = -a_{\mathrm{qs}} V_x \boldsymbol{e}_x + a'_{\mathrm{qs}} V_y \boldsymbol{e}_y \tag{47}$$

while the time constants are modelled as:

$$\tau_1 = \frac{1.1}{1 - 1.3\min(\overline{a}, 0.5)} \frac{R}{\overline{U}_0}, \qquad \tau_2 = \left[0.39 - 0.26\left(\frac{r}{R}\right)^2\right]\tau_1 \tag{48}$$

with $\overline{a}$ the mean axial induction over the rotor and $\overline{U}_0$ is the mean free stream velocity over the rotor[3]

[3] In Bladed 4.7, $\overline{U}_0$ is estimated using the annulus at 70% radius. In AeroDyn15, the mean "over the rotor" is actually computed from the values at each blade node.

Equations 45-46 may be rewritten into a single differential equation as:

$$\boldsymbol{W}_{\text{qs}} + k\tau_1 \frac{\mathrm{d}\boldsymbol{W}_{\text{qs}}}{\mathrm{d}t} = \boldsymbol{W} + \left(\tau_1 + \tau_2 + \tau_1 \frac{\mathrm{d}\tau_2}{\mathrm{d}t}\right) \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t} + \tau_1\tau_2 \frac{\mathrm{d}^2\boldsymbol{W}}{\mathrm{d}t^2} \tag{49}$$

The above may be rewritten as:

$$\begin{bmatrix} \dot{\boldsymbol{W}} \\ \ddot{\boldsymbol{W}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\frac{1}{\tau_1\tau_2}\boldsymbol{I}_2 & -\frac{1}{\tau_1\tau_2}(\tau_1 + \tau_2 + \tau_1\dot{\tau}_2)\boldsymbol{I}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \dot{\boldsymbol{W}} \end{bmatrix} + \frac{1}{\tau_1\tau_2} \begin{bmatrix} 0 & 0 \\ 1 & k\tau_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{W}_{\text{qs}} \\ \dot{\boldsymbol{W}}_{\text{qs}} \end{bmatrix} \tag{50}$$

B.2 Implementation 1: Finite difference formulation

The numerical resolution of Equations 45-46 may be done in different ways. The method used in AeroDyn [6] and the method of Hansen [5] are given below. In both methods, the term $\dot{\boldsymbol{W}}_{\text{qs}}$ is approximated using finite differences of the values of the induced velocities at t and $t + dt$. For convenience, the RHS of 46 is written $\boldsymbol{H}(t)$:

$$\boldsymbol{H}(t) \triangleq \boldsymbol{W}_{\text{qs}} + k\tau_1 \frac{\mathrm{d}\boldsymbol{W}_{\text{qs}}}{\mathrm{d}t} \tag{51}$$

In the method of Hansen, this term is assumed to be constant between two time step, while a linear variation is assumed in the AeroDyn implementation.

AeroDyn implementation The following approach is followed in AeroDyn (see [6]). First, the term $\boldsymbol{W}_{\text{qs}}$ is assumed to vary linearly between t and $t+dt$, based on the estimated derivative, then, the derivative $v\dot{W}_{\text{qs}}$ is evaluated using backward differences as follows:

$$\boldsymbol{H}(t') \approx \boldsymbol{W}^i_{\text{qs}} + t' \frac{\mathrm{d}\boldsymbol{W}_{\text{qs}}}{\mathrm{d}t} + k\tau_1 \frac{\mathrm{d}\boldsymbol{W}_{\text{qs}}}{\mathrm{d}t} \approx \boldsymbol{A} + \boldsymbol{B}t', \qquad t' \in [0, \Delta t] \tag{52}$$

$$\text{with} \qquad \boldsymbol{A} = \boldsymbol{W}^i_{\text{qs}} + \boldsymbol{B}k\tau_1, \qquad \boldsymbol{B} = \frac{\boldsymbol{W}^{i+1}_{\text{qs}} - \boldsymbol{W}^i_{\text{qs}}}{\Delta t} \tag{53}$$

Equation 45 is then integrated using Equation 52 as RHS, assuming τ_1 constant in the interval, and using the general integration formula from Equation 98, to give:

$$\boldsymbol{W}_{\text{int}}(t') = \boldsymbol{A} + \boldsymbol{B}(t' - \tau_1) + \boldsymbol{C}_0 e^{t'/\tau_1}, \qquad \text{with} \quad \boldsymbol{C}_0 = \boldsymbol{W}^i_{\text{qs}} - \boldsymbol{A} + \boldsymbol{B}\tau_1 \tag{54}$$

The term $\boldsymbol{W}_{\text{int}}(t')$ is then used in the RHS of Equation 46, which is integrated using Equation 98, assuming τ_2 constant in the interval, giving:

$$\boldsymbol{W}(t') = \boldsymbol{C}_{0,2} e^{-t'/\tau_2} + \boldsymbol{A} + \boldsymbol{B}(t' - \tau_1 - \tau_2) + \frac{\boldsymbol{C}_0}{1 - k_\tau} e^{-t'/\tau_1} \tag{55}$$

$$\text{with} \quad \boldsymbol{C}_{0,2} = \boldsymbol{W}^i_{\text{int}} - \boldsymbol{A} + \boldsymbol{B}(\tau_1 + \tau_2) - \frac{C_0}{1 - k_\tau}$$

Equation 55 is expressed at $t' = \Delta t$ to obtain $\boldsymbol{W}^{i+1}$.

Method of Hansen The numerical resolution of Equations 45-46 is presented as follows by Hansen. The term involving $\dot{W}_{\text{qs}}$ is evaluated using backward differences, and the $\boldsymbol{W}_{\text{qs}}$ is assumed to be constant in the interval (taking its final value at $t+dt$):

$$\boldsymbol{H} \approx \boldsymbol{W}_{\text{qs}}^{i+1} + k\tau_1(\boldsymbol{W}_{\text{qs}}^{i+1} - \boldsymbol{W}_{\text{qs}}^{i})/\Delta t \tag{56}$$

where the upper script i and $i+1$ represent two successive times separated by Δt. The term $\boldsymbol{H}$ is assumed constant, so that Equation 45 and Equation 46 can be successively integrated using Equation 98, leading:

$$\boldsymbol{W}_{\text{int}}^{i+1} = \boldsymbol{H} + (\boldsymbol{W}_{\text{int}}^{i} - \boldsymbol{H})e^{-\frac{\Delta t}{\tau_1}}, \qquad \boldsymbol{W}^{i+1} = \boldsymbol{W}_{\text{int}}^{i+1} + (\boldsymbol{W}^{i} - \boldsymbol{W}_{\text{int}}^{i+1})e^{-\frac{\Delta t}{\tau_2}} \tag{57}$$

B.3 Implementation 2: State-space formulation

To allow a full state space formulation and ease the linearization, the following approximations are made:

- $\dot{\tau}_2 = 0$: this approximation seem justified since the models were likely tuned at constant operating conditions
- $\dot{W}_{\text{qs}}$ can be approximated using knowledge of the BEM algorithm and ad-hoc models (see below)

The state space formulation is then:

$$\begin{bmatrix}\dot{\boldsymbol{W}}\\ \ddot{\boldsymbol{W}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_2\\ -\frac{1}{\tau_1\tau_2}\boldsymbol{I}_2 & -\frac{1}{\tau_1\tau_2}(\tau_1+\tau_2)\boldsymbol{I}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{W}\\ \dot{\boldsymbol{W}}\end{bmatrix} + \frac{1}{\tau_1\tau_2}\begin{bmatrix}0 & 0\\ 1 & k\tau_1\end{bmatrix}\begin{bmatrix}\boldsymbol{W}_{\text{qs}}\\ \dot{\boldsymbol{W}}_{\text{qs}}\end{bmatrix} \tag{58}$$