Development of a Pretwisted Beam Finite Element

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1 Introduction

This paper presents the development of a stiffness matrix for a pretwisted beam element. In a pretwisted beam, the principle axes of a cross section rotate along the beam's length. As will be shown, this pretwist leads to a coupling of bending in both planes.

The study of pretwisted beams is important when examining rotor blades (which are manufactured with a pretwisted spar) or when examining manufacturing imperfections of prismatic beams (i.e., warpage). The former is the motivation for this study, as the author is engaged in structural dynamics research of wind turbines at the National Renewable Energy Laboratory’s (NREL’s) National Wind Technology Center (NWTC).

The derived beam element is comprised of eight degrees of freedom (DOFs): two lateral deflections and two rotations at each end of the beam. In order to simplify the resulting beam element, axial deflection, torsion, and the effects of shear deformation are ignored in this derivation. The pretwisted beam element is tested for cases of static deflection of a cantilevered beam. Also, the limiting cases of the pretwisted beam (no pretwist or both principle inertias equal) are shown consistent to classical Bernoulli-Euler beam theory.

2 Governing Equations

This section presents the derivation of the differential equations governing the deflection of a pretwisted beam under distributed lateral loading. These equations are used in Section 3 as a basis for deriving the stiffness matrix of a pretwisted beam finite element. The derivation procedure used in this section follows a similar process to that presented in [1].

2.1 Assumptions

The governing equations of a pretwisted beam are derived using an extension of the classical Bernoulli-Euler beam theory. Fundamental to this theory is the assumption that plane sections normal to the beam axis prior to bending remain plane and normal to the beam axis after bending. This assumption ensures that there is no shear deformation (contrary to the fact that lateral loads, and hence cross sectional shearing forces, are present). Consequently, the strain energy of the beam accounts only for bending moment deformation. Bernoulli-Euler beam theory also ensures that transverse deflections and rotations are so small that linearization applies.

Additionally, it is assumed that the beam is fabricated with a homogenous, isotropic, and linear elastic material, that the beam axis (line of centroids) is initially straight prior to loading, and that all cross sections of the beam are identical along the length of the beam (i.e., the beam is prismatic other than the pretwist). Also, it is assumed that all lateral loading passes through the shear center of the cross sections to prevent torsion of the beam and that no torsional moments or extensional forces act on the beam. In other words, all deflections are due to bending, and loads only cause bending. Finally, it is assumed that the beam has a *uniform* rate of pretwist.

This final assumption that the pretwist rate be uniform is not as restrictive as it may first sound. The governing equations derived in this section apply only to the domain of a *single* pretwisted beam element as derived in Section 3. Thus, any form of pretwist can be achieved by axially connecting a sufficiently large number of elements together.

2.2 Geometry

Consider the geometry of an undeflected pretwisted beam, as shown in Figure 1. X,Y,Z represents a global Cartesian coordinate system, where the Z lies along the line of centroids of the undeflected pretwisted beam, which is called the beam axis. x,y,z represents a local Cartesian coordinate system aligned with the principle axes of a cross section of the undeflected pretwisted beam at location Z along the length of the beam. The origin of the global coordinate system is labeled O and the origin of the centroid of any cross section is labeled S. The orientation of the principle axes x,y relative to the global axes X,Y is given by a uniform pretwist distribution, $\phi(z)$, as follows:

$$\phi(z) = kz + \phi_O \tag{1}$$

where k is the uniform (constant) rate of twist and ϕ_O is the orientation of the principle axes of the cross section at O, the origin of the global coordinate system.

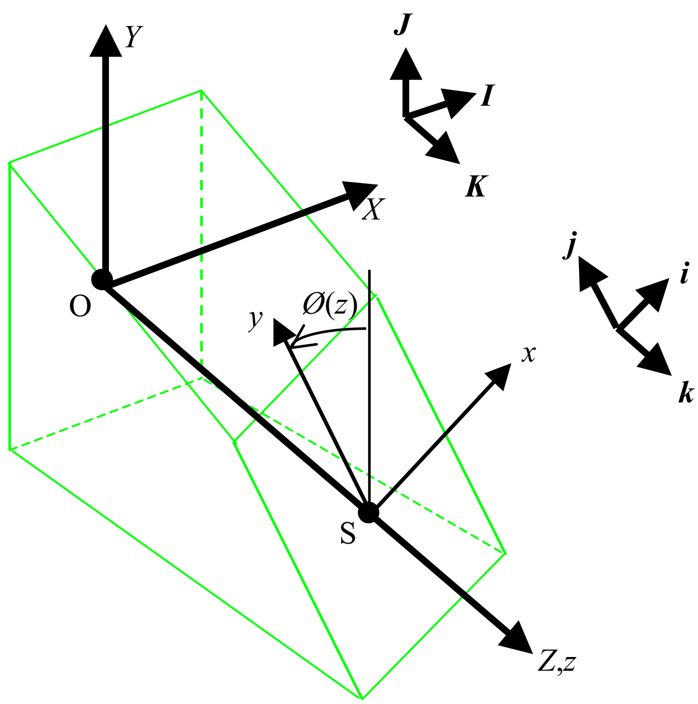


Figure 1: Pretwisted Beam Geometry

Any vector can be represented in the global unit vector triad ***I***,***J***,***K***, where each unit vector is parallel to its corresponding global axis (i.e., ***I*** is parallel to X, etc…). Likewise, any vector can be represented in the local unit vector triad $\boldsymbol{i}$,$\boldsymbol{j}$,$\boldsymbol{k}$, where each unit vector is parallel to its corresponding local axis (i.e., $\boldsymbol{i}$ is parallel to x, etc…). The relationship between the local and global vector triads is purely geometrical:

$$\begin{Bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{Bmatrix} = \begin{bmatrix} \cos\left[\phi(z)\right] & \sin\left[\phi(z)\right] & 0 \\ -\sin\left[\phi(z)\right] & \cos\left[\phi(z)\right] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{I} \\ \boldsymbol{J} \\ \boldsymbol{K} \end{Bmatrix} \tag{2}$$

or inversely:

$$\begin{Bmatrix} \boldsymbol{I} \\ \boldsymbol{J} \\ \boldsymbol{K} \end{Bmatrix} = \begin{bmatrix} \cos\left[\phi(z)\right] & -\sin\left[\phi(z)\right] & 0 \\ \sin\left[\phi(z)\right] & \cos\left[\phi(z)\right] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{Bmatrix}. \tag{3}$$

Of course, these transformation equations don't just apply to the unit vectors; they also apply to any transformation from global to local or local to global coordinates, respectively. For example, the transformation from the x,y,z system to the X,Y,Z system is simply [using Eqn. (2)]:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\left[\phi(z)\right] & \sin\left[\phi(z)\right] & 0 \\ -\sin\left[\phi(z)\right] & \cos\left[\phi(z)\right] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}. \tag{4}$$

The rate of change of the local unit vector along Z (or z) is useful in future derivations. Using Eqns. (2) and (1), the rates are as follows:

$$\begin{Bmatrix} \dfrac{d\boldsymbol{i}}{dz} \\ \dfrac{d\boldsymbol{j}}{dz} \\ \dfrac{d\boldsymbol{k}}{dz} \end{Bmatrix} = \begin{bmatrix} -k\sin\left[\phi(z)\right] & k\cos\left[\phi(z)\right] & 0 \\ -k\cos\left[\phi(z)\right] & -k\sin\left[\phi(z)\right] & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{I} \\ \boldsymbol{J} \\ \boldsymbol{K} \end{Bmatrix}. \tag{5}$$

Simplifying with the help of Eqn. (2) yields:

$$\frac{d\boldsymbol{i}}{dz} = k\boldsymbol{j} \quad , \quad \frac{d\boldsymbol{j}}{dz} = -k\boldsymbol{i} \quad , \text{ and } \quad \frac{d\boldsymbol{k}}{dz} = \boldsymbol{0}. \tag{6}$$

2.3 Deformations and Strains

When the pretwisted beam is deflected, the position vector of centroid S at z relative to the origin, $\boldsymbol{r}^{\boldsymbol{OS}}(z)$, and expressed in local coordinates, is given by the equation:

$$\boldsymbol{r}^{OS}(z) = u(z)\boldsymbol{i} + v(z)\boldsymbol{j} + z\boldsymbol{k}\,, \tag{7}$$

where $u(z)$ and $v(z)$ represent the lateral deflection of the beam's centroid expressed in the local x- and y- directions respectively. With the help of Eqn. (6), the derivative of this position vector with respect to z is as follows:

$$\frac{d\boldsymbol{r}^{os}(z)}{dz}=\left[\frac{du(z)}{dz}-kv(z)\right]\boldsymbol{i}+\left[\frac{dv(z)}{dz}+ku(z)\right]\boldsymbol{j}+\boldsymbol{k}\cdot \tag{8}$$

Recall the assumption that plane sections normal to the beam axis prior to bending remain plane and normal to the beam axis after bending. It fallows that the rotations of the beam's cross section about the x- and y- axes after bending, $\theta_x(z)$ and $\theta_y(z)$ respectively, are given by:

$$\theta_x(z)=-\frac{d\boldsymbol{r}^{os}(r)}{dz}\bullet\boldsymbol{j} \quad \text{and} \quad \theta_y(z)=\frac{d\boldsymbol{r}^{os}(r)}{dz}\bullet\boldsymbol{i}\cdot \tag{9}$$

The negative sign is built into the equation for $\theta_x(z)$ since a positive change in $\boldsymbol{r}^{OS}(z)$ in the y-direction correlates with a negative rotation about the x-axis. With the help of Eqn. (8) these become:

$$\theta_x(z)=-\frac{dv(z)}{dz}-ku(z) \quad \text{and} \quad \theta_y(z)=\frac{du(z)}{dz}-kv(z)\cdot \tag{10}$$

Since plane sections remain plane after bending, the axial deflection of any point x,y in the cross section at z, $W(x,y,z)$, is directly proportional to these rotations as follows:

$$W(x,y,z)=y\theta_x(z)-x\theta_y(z)\cdot \tag{11}$$

The three components of deflection (two lateral and one axial) of any point X,Y,Z may now be expressed in global coordinates with the help of Eqns. (3), (4), and (11) as follows:

$$U(Z)=u(Z)\cos\left[\phi(Z)\right]-v(Z)\sin\left[\phi(Z)\right] \tag{12}$$

$$V(Z)=u(Z)\sin\left[\phi(Z)\right]+v(Z)\cos\left[\phi(Z)\right] \tag{13}$$

and

$$W(X,Y,Z)=\left\{-X\sin\left[\phi(Z)\right]+Y\cos\left[\phi(Z)\right]\right\}\theta_x(Z)-\left\{X\cos\left[\phi(Z)\right]+Y\sin\left[\phi(Z)\right]\right\}\theta_y(Z) \tag{14}$$

where $U(Z)$ and $V(Z)$ represent the lateral deflection of the beam's centroid expressed in the global X- and Y- directions respectively.

Expressing the deflections in global coordinates is useful for determining the strain components. Clearly,

$$\varepsilon_{XX}=\frac{\partial U}{\partial X}=0 \quad , \quad \varepsilon_{YY}=\frac{\partial V}{\partial Y}=0 \quad , \text{ and } \quad \gamma_{XY}=\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}=0 \tag{15}$$

Also,

$$
\begin{aligned}
\gamma_{XZ} &= \frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} \\
&= \frac{du(Z)}{dZ}\cos\left[\phi(Z)\right] - ku(z)\sin\left[\phi(Z)\right] - \frac{dv(Z)}{dZ}\sin\left[\phi(Z)\right] - kv(Z)\cos\left[\phi(Z)\right] \\
&+ \frac{dv(z)}{dz}\sin\left[\phi(Z)\right] + ku(z)\sin\left[\phi(Z)\right] - \frac{du(z)}{dz}\cos\left[\phi(Z)\right] + kv(z)\cos\left[\phi(Z)\right] \\
&= 0
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
\gamma_{YZ} &= \frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} \\
&= \frac{du(Z)}{dZ}\sin\left[\phi(Z)\right] + ku(z)\cos\left[\phi(Z)\right] + \frac{dv(Z)}{dZ}\cos\left[\phi(Z)\right] - kv(Z)\sin\left[\phi(Z)\right] \\
&- \frac{dv(z)}{dz}\cos\left[\phi(Z)\right] - ku(z)\cos\left[\phi(Z)\right] - \frac{du(z)}{dz}\sin\left[\phi(Z)\right] + kv(z)\sin\left[\phi(Z)\right] \\
&= 0
\end{aligned} \tag{17}
$$

In fact, the only nonzero strain is the axial strain, ε_{ZZ}, which is consistent with classical Bernoulli-Euler beam theory:

$$
\begin{aligned}
\varepsilon_{ZZ} &= \frac{\partial W}{\partial Z} \\
&= k\left\{X\cos\left[\phi(Z)\right] + Y\sin\left[\phi(Z)\right]\right\}\left[\frac{dv(z)}{dz} + ku(z)\right] - \left\{-X\sin\left[\phi(Z)\right] + Y\cos\left[\phi(Z)\right]\right\}\left[\frac{d^2v(z)}{dz^2} + k\frac{du(z)}{dz}\right] \\
&- k\left\{-X\sin\left[\phi(Z)\right] + Y\cos\left[\phi(Z)\right]\right\}\left[\frac{du(z)}{dz} - kv(z)\right] - \left\{X\cos\left[\phi(Z)\right] + Y\sin\left[\phi(Z)\right]\right\}\left[\frac{d^2u(z)}{dz^2} - k\frac{dv(z)}{dz}\right]
\end{aligned} \tag{18}
$$

With the help of Eqn. (4), Eqn. (18) simplifies to:

$$
\varepsilon_{ZZ} = -x\left[\frac{d^2u(z)}{dz^2} - 2k\frac{dv(z)}{dz} - k^2u(z)\right] - y\left[\frac{d^2v(z)}{dz^2} + 2k\frac{du(z)}{dz} - k^2v(z)\right] \tag{19}
$$

2.4 Constitutive Equations

Since the axial strain is the only nonzero strain, the strain energy of the beam, $\mathcal{U}$, can be easily expressed as:

$$
\mathcal{U} = \frac{1}{2}\int_V \sigma_{ZZ}\varepsilon_{ZZ}\,dV = \frac{E}{2}\int_V \varepsilon_{ZZ}^2\,dV = \frac{E}{2}\int_0^L\left(\int_A \varepsilon_{ZZ}^2\,dA\right)dz \tag{20}
$$

where σ_{ZZ} is the normal stress, E is Young's Modulus of Elasticity, and V, A, and L are the volume, cross sectional area, and length of the beam respectively. Inserting Eqn. (19) into Eqn. (20) yields:

$$\mathcal{U}=\frac{E}{2}\int_0^L\left\{\begin{array}{l}\left[\frac{d^2u(z)}{dz^2}-2k\frac{dv(z)}{dz}-k^2u(z)\right]^2\int_A x^2dA+\left[\frac{d^2v(z)}{dz^2}+2k\frac{du(z)}{dz}-k^2v(z)\right]^2\int_A y^2dA\\ +\left[\frac{d^2u(z)}{dz^2}-2k\frac{dv(z)}{dz}-k^2u(z)\right]\left[\frac{d^2v(z)}{dz^2}+2k\frac{du(z)}{dz}-k^2v(z)\right]\int_A xydA\end{array}\right\}dz\,. \tag{21}$$

By noting the definitions of the principle moments of inertia, I_{xx} and I_{yy}, about the principle x- and y- axes respectively:

$$I_{xx}=\int_A y^2dA\ ,\quad I_{yy}=\int_A x^2dA\ ,\ \text{and}\quad I_{xy}=\int_A xydA=0 \tag{22}$$

the strain energy simplifies to:

$$\mathcal{U}=\frac{EI_{xx}}{2}\int_0^L\left[\frac{d^2v(z)}{dz^2}+2k\frac{du(z)}{dz}-k^2v(z)\right]^2dz+\frac{EI_{yy}}{2}\int_0^L\left[\frac{d^2u(z)}{dz^2}-2k\frac{dv(z)}{dz}-k^2u(z)\right]^2dz \tag{23}$$

Of course, the strain energy can also be written in terms of the internal bending moments about the x- and y- axes, $M_x(z)$ and $M_y(z)$ respectively, as follows:

$$\mathcal{U}=\frac{1}{2}\int_0^L\frac{M_x(z)^2}{EI_{xx}}dz+\frac{1}{2}\int_0^L\frac{M_y(z)^2}{EI_{yy}}dz \tag{24}$$

Comparing Eqn. (24) with Eqn. (23), it is seen that the internal bending moment constitutive equations are as follows:

$$M_x(z)=-EI_{xx}\left[\frac{d^2v(z)}{dz^2}+2k\frac{du(z)}{dz}-k^2v(z)\right] \tag{25}$$

and

$$M_y(z)=EI_{yy}\left[\frac{d^2u(z)}{dz^2}-2k\frac{dv(z)}{dz}-k^2u(z)\right] \tag{26}$$

The minus sign is built into Eqn. (25) since a positive moment about the x-axis produces a negative curvature in the y-direction.

2.5 Equilibrium Equations

The equilibrium equations for the pretwisted beam are obtained from a simple static analysis of a differential element of the beam. Consider the free body diagram of the differential pretwisted beam element of length dz and pretwist $d\phi$ as shown in Figure 2. The differential pretwisted beam element is loaded by distributed lateral forces (per length) in the x- and y-directions, $q_x(z)$ and $q_y(z)$ respectively, internal shear forces in the x- and y-directions, $V_x(z)$ and $V_y(z)$ respectively, and internal bending moments in the x- and y-directions, $M_x(z)$ and $M_y(z)$ respectively. The six standard static equilibrium equations are:

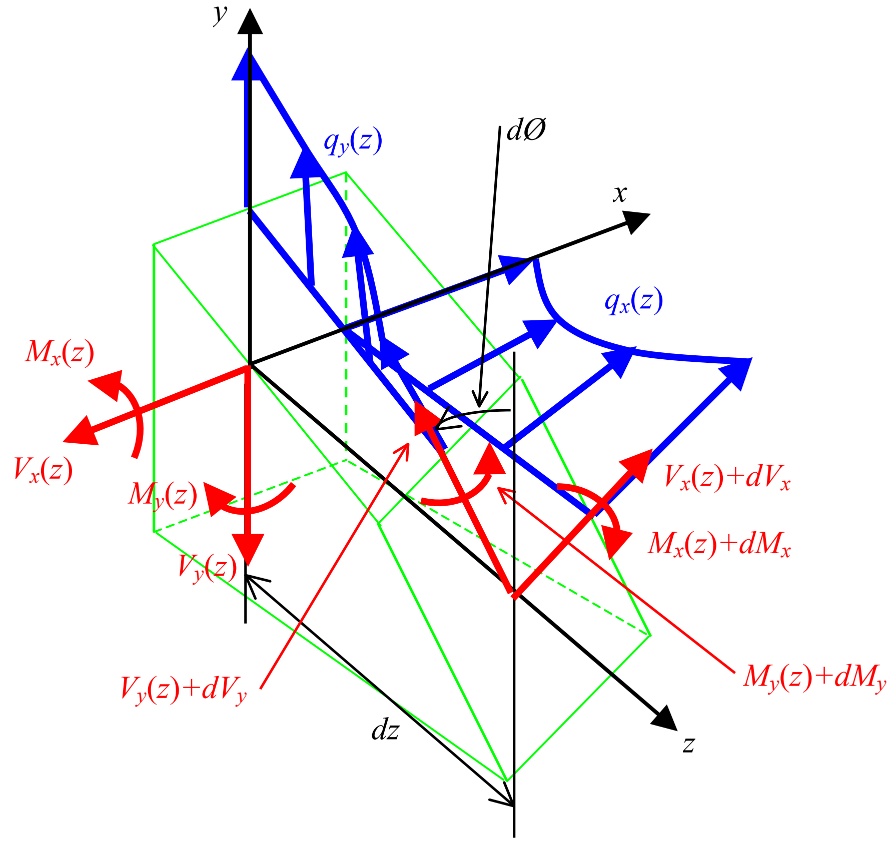


Figure 2: Pretwisted Beam Free Body Diagram

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \tag{27}$$

From Figure 2 it is obvious that the sum of forces along and the sum of the moments about the *z*-axis are identically satisfied. The sum of the forces in the *x*- and *y*-directions are as follows:

$$\sum F_x = 0 = \left[V_x(z) + dV_x\right]\cos\left[d\phi\right] - \left[V_y(z) + dV_y\right]\sin\left[d\phi\right] - V_x(z) + q_x(z)\,dz \tag{28}$$

and

$$\sum F_y = 0 = \left[V_x(z) + dV_x\right]\sin\left[d\phi\right] + \left[V_y(z) + dV_y\right]\cos\left[d\phi\right] - V_y(z) + q_y(z)\,dz\,. \tag{29}$$

Since the differential element is small, the differential rotation, *dø*, must also be small. Making use of the small angle approximations (cos[*dø*] = 1 and sin[*dø*] = *dø*) and ignoring all products of differentials, these equations simplify to:

$$q_x(z) = -\frac{dV_x(z)}{dz} + kV_y(z) \tag{30}$$

and

$$q_y(z) = -\frac{dV_y(z)}{dz} - kV_x(z) \tag{31}$$

where the substitution:

$$\frac{d\phi}{dz} = k, \tag{32}$$

which is a direct result of Eqn. (1), has been made.

Similarly, the sum of the moments about the *x*- and *y*-axes are as follows:

$$\begin{aligned}\sum M_x = 0 = &\left[M_x(z) + dM_x\right]\cos[d\phi] - \left[M_y(z) + dM_y\right]\sin[d\phi] - M_x(z) \\ &-\left[V_x(z) + dV_x\right]\sin[d\phi]dz - \left[V_y(z) + dV_y\right]\cos[d\phi]dz - q_y(z)dz\frac{dz}{2}\end{aligned} \tag{33}$$

and

$$\begin{aligned}\sum M_y = 0 = &\left[M_x(z) + dM_x\right]\sin[d\phi] + \left[M_y(z) + dM_y\right]\cos[d\phi] - M_y(z) \\ &+\left[V_x(z) + dV_x\right]\cos[d\phi]dz - \left[V_y(z) + dV_y\right]\sin[d\phi]dz + q_x(z)dz\frac{dz}{2}\end{aligned}. \tag{34}$$

With the help of Eqn. (32) and the small angle and small product approximations, Eqn. (33) and Eqn. (34) simplify to:

$$V_y(z) = \frac{dM_x(z)}{dz} - kM_y(z) \tag{35}$$

and

$$V_x(z) = -\frac{dM_y(z)}{dz} - kM_x(z) \cdot \tag{36}$$

2.6 Overall Governing Equations

The following two coupled fourth order linear differential equations, which characterize the deflection of a pretwisted beam in local principle coordinates under distributed lateral loading, are derived by substituting the constitutive equations [Eqn. (25) and Eqn. (26)] into the shear force equilibrium equations [Eqn. (35) and Eqn. (36)] and substituting the resulting equations into the distributed lateral load equilibrium equations [Eqn. (30) and Eqn. (31)]:

$$\begin{aligned}q_x(z) = &EI_{yy}\left[\frac{d^4u(z)}{dz^4} - 2k\frac{d^3v(z)}{dz^3} - 2k^2\frac{d^2u(z)}{dz^2} + 2k^3\frac{dv(z)}{dz} + k^4u(z)\right] \\ &- EI_{xx}\left[2k\frac{d^3v(z)}{dz^3} + 4k^2\frac{d^2u(z)}{dz^2} - 2k^3\frac{dv(z)}{dz}\right]\end{aligned} \tag{37}$$

and

$$\begin{aligned}q_y(z) = &EI_{xx}\left[\frac{d^4v(z)}{dz^4} + 2k\frac{d^3u(z)}{dz^3} - 2k^2\frac{d^2v(z)}{dz^2} - 2k^3\frac{du(z)}{dz} + k^4v(z)\right] \\ &+ EI_{yy}\left[2k\frac{d^3u(z)}{dz^3} - 4k^2\frac{d^2v(z)}{dz^2} - 2k^3\frac{du(z)}{dz}\right]\end{aligned}. \tag{38}$$

3 Pretwisted Beam Finite Elements

The governing equations that characterize the deflection of a pretwisted beam under distributed lateral loading, as derived in Section 2 [see Eqn. (37) and Eqn. (38)], are a coupled pair of fourth order differential equations. Because these equations would be very difficult to solve analytically, even under the simplest loading conditions, numerical solutions using the finite element method (FEM) are necessary. As such, this section presents the derivation of the stiffness matrix of pretwisted beam finite element. The procedure very closely follows the well-known method of developing the stiffness matrix for plane bending of unpretwisted beams.

3.1 Assumptions

The governing equations that characterize the deflection of a pretwisted beam under distributed lateral loading, as derived in Section 2, are used as a basis for deriving the stiffness matrix of a pretwisted beam finite element in this Section. As a natural result, the same assumptions from which the pretwisted beam governing equations were derived are also applicable for the pretwisted beam finite elements (see Section 2.1).

However, one additional assumption must be made in order to derive a linear pretwisted beam finite element. This is the assumption that *the overall pretwist across each element is small*. The necessity of this assumption will be evident in the derivations of Section 3.2. This constraint is not as restrictive as it first may sound. If, for example, a beam has a large overall pretwist, say 90º, all this means is that the beam must be split into a number of different elements, say 30, so that the overall pretwist across each single element is small.

Letting ε be the overall pretwist across an element of length L, this constraint can be stated mathematically as follows:

$$\varepsilon = kL \ll 1 \tag{39}$$

or after rearrangement:

$$k = \frac{\varepsilon}{L} \ll \frac{1}{L} \tag{40}$$

If it is also ensured that the element is not infinitesimally small (i.e., L is a reasonable length), the assumption that the overall pretwist across each element is small implies that the uniform rate of pretwist across each element must also is small. With this assumption, all products of the uniform rate of pretwist k can be neglected (i.e., $k^2 \approx 0$, etc…), and the governing equations derived in Section 2, Eqn. (37) and Eqn. (38), simplify to:

$$q_x(z) = EI_{yy}\left[\frac{d^4u(z)}{dz^4} - 2k\frac{d^3v(z)}{dz^3}\right] - EI_{xx}\left[2k\frac{d^3v(z)}{dz^3}\right] \tag{41}$$

and

$$q_y(z) = EI_{xx}\left[\frac{d^4 v(z)}{dz^4} + 2k\frac{d^3 u(z)}{dz^3}\right] + EI_{yy}\left[2k\frac{d^3 u(z)}{dz^3}\right]. \tag{42}$$

This is the form of the governing equations for which the pretwisted beam finite element is derived.

3.2 Degrees of Freedom and Shape Functions

Following the development of finite elements for plane bending, it is obvious that the pretwisted beam finite element will have two nodes, one on each end of the beam, and that each bending direction requires four DOFs—two at each node. Since it is assumed that only bending deflections are considered and that there are two bending directions, the pretwisted beam finite element is seen to have a total of eight DOFs—two translational deflections, and two rotations at each node. The ordering of DOFs are chosen by the authors discretion and are depicted in Figure 3. The components q_1 to q_4 correspond to bending in the local y-direction and components q_5 to q_8 correspond to bending in the local x-direction.

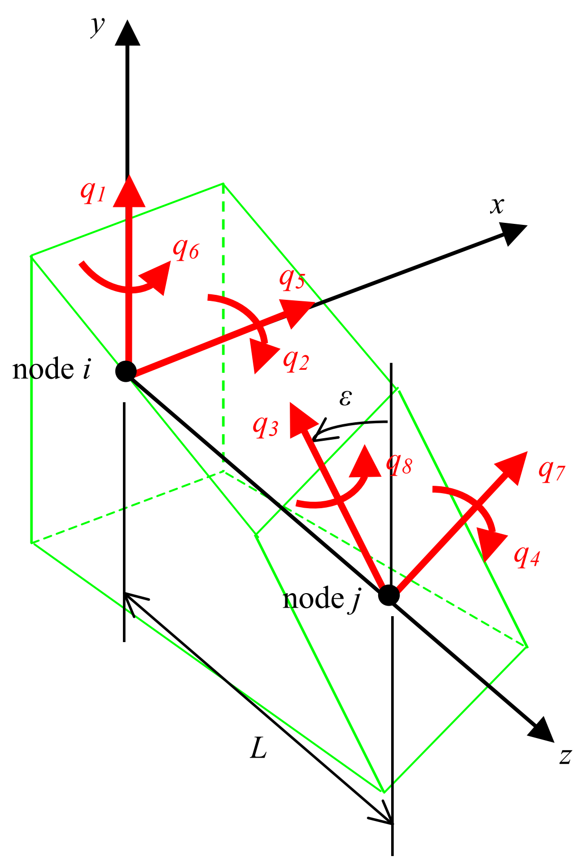


Figure 3: Pretwisted Beam Finite Element DOFs

Letting $N_{ui}(z)$ and $N_{vi}(z)$ be the shape functions corresponding to deflection in the local x- and y-directions respectively due to DOF q_i, it is seen that the displacement field may be written in terms of the following interpolation:

$$\begin{Bmatrix} u(z) \\ v(z) \end{Bmatrix} = \begin{bmatrix} N_{u1}(z) & N_{u2}(z) & N_{u3}(z) & N_{u4}(z) & N_{u5}(z) & N_{u6}(z) & N_{u7}(z) & N_{u8}(z) \\ N_{v1}(z) & N_{v2}(z) & N_{v3}(z) & N_{v4}(z) & N_{v5}(z) & N_{v6}(z) & N_{v7}(z) & N_{v8}(z) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} \tag{43}$$

or,

$$\begin{Bmatrix} u(z) \\ v(z) \end{Bmatrix} = \begin{bmatrix} \lfloor N_u(z) \rfloor \\ \lfloor N_v(z) \rfloor \end{bmatrix} \{q\} \tag{44}$$

where,

$$\lfloor N_u(z) \rfloor = \lfloor N_{u1}(z) \quad N_{u2}(z) \quad N_{u3}(z) \quad N_{u4}(z) \quad N_{u5}(z) \quad N_{u6}(z) \quad N_{u7}(z) \quad N_{u8}(z) \rfloor \tag{45}$$

$$\lfloor N_v(z) \rfloor = \lfloor N_{v1}(z) \quad N_{v2}(z) \quad N_{v3}(z) \quad N_{v4}(z) \quad N_{v5}(z) \quad N_{v6}(z) \quad N_{v7}(z) \quad N_{v8}(z) \rfloor \tag{46}$$

and

$$\{q\} = \lfloor q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 \quad q_7 \quad q_8 \rfloor^T \tag{47}$$

Using a polynomial form for the shape functions, the polynomials must be fourth order if the displacement fields are to satisfy the governing equations [see Eqn. (41) and Eqn. (42)]. Written in terms of the dimensionless "natural" coordinate,

$$\xi(z) = \frac{z}{L}, \tag{48}$$

the generic shape functions $N_{ui}(z)$ and $N_{vi}(z)$ can thus be written as:

$$N_{ui}(z) = a_{0i} + a_{1i}\xi + a_{2i}\xi^2 + a_{3i}\xi^3 + a_{4i}\xi^4 \tag{49}$$

and

$$N_{vi}(z) = b_{0i} + b_{1i}\xi + b_{2i}\xi^2 + b_{3i}\xi^3 + b_{4i}\xi^4 . \tag{50}$$

The first four derivatives of these shape functions with respect to the axial z-direction are needed in the subsequent analysis and are provided here for convenience. The first derivatives of $N_{ui}(z)$ and $N_{vi}(z)$ with respect to z are:

$$\frac{dN_{ui}(z)}{dz} = \frac{1}{L}\left(a_{1i} + 2a_{2i}\xi + 3a_{3i}\xi^2 + 4a_{4i}\xi^3\right) \tag{51}$$

and

$$\frac{dN_{vi}(z)}{dz} = \frac{1}{L}\left(b_{1i} + 2b_{2i}\xi + 3b_{3i}\xi^2 + 4b_{4i}\xi^3\right) . \tag{52}$$

The second derivatives of $N_{ui}(z)$ and $N_{vi}(z)$ with respect to z are:

$$\frac{d^2 N_{ui}(z)}{dz^2} = \frac{1}{L^2}\left(2a_{2i} + 6a_{3i}\xi + 12a_{4i}\xi^2\right) \tag{53}$$

and

$$\frac{d^2 N_{vi}(z)}{dz^2} = \frac{1}{L^2}\left(2b_{2i} + 6b_{3i}\xi + 12b_{4i}\xi^2\right). \tag{54}$$

The third derivatives of $N_{ui}(z)$ and $N_{vi}(z)$ with respect to z are:

$$\frac{d^3 N_{ui}(z)}{dz^3} = \frac{1}{L^3}\left(6a_{3i} + 24a_{4i}\xi\right) \tag{55}$$

and

$$\frac{d^3 N_{vi}(z)}{dz^3} = \frac{1}{L^3}\left(6b_{3i} + 24b_{4i}\xi\right). \tag{56}$$

And finally, the fourth derivatives of $N_{ui}(z)$ and $N_{vi}(z)$ with respect to z are:

$$\frac{d^4 N_{ui}(z)}{dz^4} = \frac{1}{L^4}\left(24a_{4i}\right) \tag{57}$$

and

$$\frac{d^4 N_{vi}(z)}{dz^4} = \frac{1}{L^4}\left(24b_{4i}\right). \tag{58}$$

The 10 unknown polynomial coefficients in the shape functions for each DOF q_i; a_{0i}, a_{1i}, …, b_{4i}; can be determined through 10 constraint equations. Since the displacement field throughout a single element must satisfy the *homogenous* form of the governing equations when q_i is unity and all other DOFs are zero, two of the constraint equations for each DOF are obtained by substituting $N_{ui}(z)$ and $N_{vi}(z)$ in for $u(z)$ and $v(z)$ in Eqn. (41) and Eqn. (42) as follows:

$$q_x(z) = 0 = EI_{yy}\left[\frac{d^4 N_{ui}(z)}{dz^4} - 2k\frac{d^3 N_{vi}(z)}{dz^3}\right] - EI_{xx}\left[2k\frac{d^3 N_{vi}(z)}{dz^3}\right] \tag{59}$$

and

$$q_y(z) = 0 = EI_{xx}\left[\frac{d^4 N_{vi}(z)}{dz^4} + 2k\frac{d^3 N_{ui}(z)}{dz^3}\right] + EI_{yy}\left[2k\frac{d^3 N_{ui}(z)}{dz^3}\right] \tag{60}$$

The homogenous form of the governing equations apply since finite elements can be loaded only at the nodes, and thus, $q_x(z)$ and $q_y(z)$ must equal zero. Substituting the polynomial forms of the shape functions into Eqn. (59) and Eqn. (60) results in the following two constraint equations for the unknown polynomial coefficients:

$$0 = EI_{yy}\left(2a_{4i} - kLb_{3i} - 4kL\xi b_{4i}\right) - EI_{xx}\left(kLb_{3i} + 4kL\xi b_{4i}\right) \tag{61}$$

and

$$0 = EI_{xx}\left(2b_{4i} + kLa_{3i} + 4kL\xi a_{4i}\right) + EI_{xx}\left(kLa_{3i} + 4kL\xi a_{4i}\right). \tag{62}$$

Eight additional constraint equations for the unknown polynomial coefficients of each DOF are obtained by noting that the displacement field at both end nodes ($z = 0$ and $z = L$ or $\xi = 0$ and $\xi = 1$) must satisfy the conditions that occur when q_i is unity and all other DOFs are zero. Using the polynomial forms of the shape functions, these eight additional constraint equations for each DOF are as follows:

$$u(0) = N_{ui}(0) = a_{0i} \tag{63}$$

$$v(0) = N_{vi}(0) = b_{0i} \tag{64}$$

$$u(L) = N_{ui}(L) = a_{0i} + a_{1i} + a_{2i} + a_{3i} + a_{4i} \tag{65}$$

$$v(L) = N_{vi}(L) = b_{0i} + b_{1i} + b_{2i} + b_{3i} + b_{4i} \tag{66}$$

$$\theta_x(0) = -\left.\frac{dN_{vi}(z)}{dz}\right|_{z=0} - kN_{ui}(0) = -\frac{1}{L}b_{1i} - ka_{0i} \tag{67}$$

$$\theta_y(0) = \left.\frac{dN_{ui}(z)}{dz}\right|_{z=0} - kN_{vi}(0) = \frac{1}{L}a_{1i} - kb_{0i} \tag{68}$$

$$\theta_x(L) = -\left.\frac{dN_{vi}(z)}{dz}\right|_{z=L} - kN_{ui}(L) = -\frac{1}{L}\left(b_{1i} + 2b_{2i} + 3b_{3i} + 4b_{4i}\right) - k\left(a_{0i} + a_{1i} + a_{2i} + a_{3i} + a_{4i}\right) \tag{69}$$

and

$$\theta_y(L) = -\left.\frac{dN_{ui}(z)}{dz}\right|_{z=L} - kN_{vi}(L) = \frac{1}{L}\left(a_{1i} + 2a_{2i} + 3a_{3i} + 4a_{4i}\right) - k\left(b_{0i} + b_{1i} + b_{2i} + b_{3i} + b_{4i}\right). \tag{70}$$

Placing Eqn. (61) through (70) in matrix form results in:

$$\begin{Bmatrix} u(0) \\ v(0) \\ u(L) \\ v(L) \\ L\theta_x(0) \\ L\theta_y(0) \\ L\theta_x(L) \\ L\theta_y(L) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ -\varepsilon & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ -\varepsilon & -\varepsilon & -\varepsilon & -\varepsilon & -\varepsilon & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 & -\varepsilon & -\varepsilon & -\varepsilon & -\varepsilon & -\varepsilon \\ 0 & 0 & 0 & 0 & 2EI_{yy} & 0 & 0 & 0 & -\varepsilon\left(EI_{xx} + EI_{yy}\right) & -4\varepsilon\xi\left(EI_{xx} + EI_{yy}\right) \\ 0 & 0 & 0 & \varepsilon\left(EI_{xx} + EI_{yy}\right) & 4\varepsilon\xi\left(EI_{xx} + EI_{yy}\right) & 0 & 0 & 0 & 0 & 2EI_{xx} \end{bmatrix} \begin{Bmatrix} a_{0i} \\ a_{1i} \\ a_{2i} \\ a_{3i} \\ a_{4i} \\ b_{0i} \\ b_{1i} \\ b_{2i} \\ b_{3i} \\ b_{4i} \end{Bmatrix} \tag{71}$$

where the substitution:

$$\varepsilon = kL \tag{72}$$

has been made.

In order to find the unknown polynomial coefficients for each DOF, the inverse of the constraint matrix must be found and the appropriate displacement boundary conditions must be specified. For example, when q_1 is unity and all other DOFs are zero, the required boundary specifications are that $v(0)$ must equal unity and $u(0)$, $u(L)$, $v(L)$, $\theta_x(0)$, $\theta_y(0)$, $\theta_x(L)$, and $\theta_y(L)$ must all be zero.

Following this process, the Mathematica code provided in Appendix A is used to find all 16 shape functions. The resulting shape functions are as follows:

$$N_{u1}(z)=\varepsilon\left[\xi+\left(\frac{EI_{xx}-EI_{yy}}{EI_{yy}}\right)\xi^2-\left(\frac{2EI_{xx}+EI_{yy}}{EI_{yy}}\right)\xi^3+\left(\frac{EI_{xx}+EI_{yy}}{EI_{yy}}\right)\xi^4\right] \tag{73}$$

$$N_{u2}(z)=\varepsilon L\left[-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{yy}}\right)\xi^2+\left(\frac{EI_{xx}+EI_{yy}}{EI_{yy}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{yy}}\right)\xi^4\right] \tag{74}$$

$$N_{u3}(z)=\varepsilon\left[-\left(\frac{EI_{xx}+2EI_{yy}}{EI_{yy}}\right)\xi^2+\left(\frac{2EI_{xx}+3EI_{yy}}{EI_{yy}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{EI_{yy}}\right)\xi^4\right] \tag{75}$$

$$N_{u4}(z)=\varepsilon L\left[-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{yy}}\right)\xi^2+\left(\frac{EI_{xx}+EI_{yy}}{EI_{yy}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{yy}}\right)\xi^4\right] \tag{76}$$

$$N_{u5}(z)=1-3\xi^2+2\xi^3 \tag{77}$$

$$N_{u6}(z)=L\left(\xi-2\xi^2+\xi^3\right) \tag{78}$$

$$N_{u7}(z)=3\xi^2-2\xi^3 \tag{79}$$

$$N_{u8}(z)=L\left(-\xi^2+\xi^3\right) \tag{80}$$

and

$$N_{v1}(z)=1-3\xi^2+2\xi^3 \tag{81}$$

$$N_{v2}(z)=L\left(-\xi+2\xi^2-\xi^3\right) \tag{82}$$

$$N_{v3}(z)=3\xi^2-2\xi^3 \tag{83}$$

$$N_{v4}(z)=L\left(\xi^2-\xi^3\right) \tag{84}$$

$$N_{v5}(z)=\varepsilon\left[-\xi+\left(\frac{EI_{xx}-EI_{yy}}{EI_{xx}}\right)\xi^2+\left(\frac{EI_{xx}+2EI_{yy}}{EI_{xx}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{EI_{xx}}\right)\xi^4\right] \tag{85}$$

$$N_{v6}(z)=\varepsilon L\left[-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{xx}}\right)\xi^2+\left(\frac{EI_{xx}+EI_{yy}}{EI_{xx}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{xx}}\right)\xi^4\right] \tag{86}$$

$$N_{v7}(z)=\varepsilon\left[\left(\frac{2EI_{xx}+EI_{yy}}{EI_{xx}}\right)\xi^2-\left(\frac{3EI_{xx}+2EI_{yy}}{EI_{xx}}\right)\xi^3+\left(\frac{EI_{xx}+EI_{yy}}{EI_{xx}}\right)\xi^4\right] \tag{87}$$

$$N_{v8}(z)=\varepsilon L\left[-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{xx}}\right)\xi^2+\left(\frac{EI_{xx}+EI_{yy}}{EI_{xx}}\right)\xi^3-\left(\frac{EI_{xx}+EI_{yy}}{2EI_{xx}}\right)\xi^4\right]. \tag{88}$$

The shape functions $N_{u5}(z)$ through $N_{u8}(z)$ and $N_{v1}(z)$ through $N_{v4}(z)$ are the standard Hermitian cubic shape functions in the natural coordinate defined by Eqn. (48) for bending in the local x- and y-directions respectively. The shape functions $N_{u1}(z)$ through $N_{u4}(z)$ and $N_{v5}(z)$ through $N_{v8}(z)$ are cross coupling shape functions, which exist because of the pretwist. These shape functions are identical to those documented in [2], though the derivational details are left out of the article.

A crosscheck for completeness is also made in Appendix A by summing up the shape functions in each direction and verifying that their totals equal unity. The sum totals turn out to be:

$$\sum_{i=1}^{8} N_{ui}(z) = 1 - \varepsilon L\left(\frac{EI_{xx} + EI_{yy}}{16EI_{yy}}\right) \tag{89}$$

and

$$\sum_{i=1}^{8} N_{vi}(z) = 1 - \varepsilon L\left(\frac{EI_{xx} + EI_{yy}}{16EI_{xx}}\right) \tag{90}$$

With $\varepsilon \ll 1$, the second terms in the right hand side of Eqn. (89) and Eqn. (90) are negligible compared to unity; thus the sum of shape functions in each direction is unity and the completeness crosscheck is satisfactory.

As an explanation on why it is necessary to employ the small overall pretwist constraint on each element, consider the case in which this constraint is not enforced. If the overall pretwist across a single element is not assumed to be small, then notice that terms involving powers of ξ would appear in Eqn. (71). With this being the case, the polynomial coefficients found upon inverting the modified Eqn. (71) would be found to be functions of ξ. In other words, the polynomial coefficients would not be constant and the assumed polynomial form of the shape functions would be incorrect. In fact, the form of the required shape functions is not known. If one does exist, it is most likely considerably more complex than the polynomial form employed above. To eradicate these problems, the constraint that the overall pretwist across a single element be small must be made.

3.3 Local Element Stiffness Matrix

The element stiffness matrix associated with the DOFs and shape functions found in the previous section may be found using the total potential energy functional method, where the element stiffness matrix, [$\bar{K}^{(e)}$], is obtained using the strain energy of the beam as follows:

$$\mathcal{U} = \frac{1}{2}\{q\}^T\left[\bar{K}^{(e)}\right]\{q\}. \tag{91}$$

The bar over the symbol K in the stiffness matrix is used to indicate that the stiffness matrix is expressed in the local x,y,z system.

Inserting the displacement field interpolation from Eqn. (44) into the strain energy formulation [Eqn. (23)] yields:

$$\begin{aligned}\mathcal{U} = {} & \frac{EI_{xx}}{2}\int_0^L\left[\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor\{q\} + 2k\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor\{q\} - k^2\left\lfloor N_v(z)\right\rfloor\{q\}\right]^2 dz \\ & + \frac{EI_{yy}}{2}\int_0^L\left[\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor\{q\} - 2k\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor\{q\} - k^2\left\lfloor N_u(z)\right\rfloor\{q\}\right]^2 dz\end{aligned}. \tag{92}$$

Upon expansion, Eqn. (92) produces:

$$\mathcal{U}=\frac{EI_{xx}}{2}\{q\}^T\int_0^L\begin{bmatrix}\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor+2k\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor-k^2\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\lfloor N_v(z)\rfloor\\+2k\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor+4k^2\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor^T\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor-2k^3\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor^T\lfloor N_v(z)\rfloor\\-k^2\lfloor N_v(z)\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor-2k^3\lfloor N_v(z)\rfloor^T\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor+k^4\lfloor N_v(z)\rfloor^T\lfloor N_v(z)\rfloor\end{bmatrix}dz\{q\}$$

$$+\frac{EI_{yy}}{2}\{q\}^T\int_0^L\begin{bmatrix}\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor-2k\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor-k^2\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\lfloor N_u(z)\rfloor\\-2k\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor+4k^2\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor^T\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor+2k^3\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor^T\lfloor N_u(z)\rfloor\\-k^2\lfloor N_u(z)\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor+2k^3\lfloor N_u(z)\rfloor^T\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor+k^4\lfloor N_u(z)\rfloor^T\lfloor N_u(z)\rfloor\end{bmatrix}dz\{q\}\quad. \qquad (93)$$

Finally, neglecting all terms involving products of k due to the small pretwist assumption, the strain energy written in terms of shape functions simplifies to:

$$\mathcal{U}=\frac{1}{2}\{q\}^T\begin{bmatrix}EI_{xx}\int_0^L\left[\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor+2k\left(\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor+\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor\right)\right]dz\\+EI_{yy}\int_0^L\left[\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor-2k\left(\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor+\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor\right)\right]dz\end{bmatrix}\{q\}\,. \qquad (94)$$

Comparison of Eqn. (94) with Eqn. (91) shows that the element stiffness matrix is defined by:

$$\begin{aligned}\left[\bar{K}^{(e)}\right]=&EI_{xx}\int_0^L\left[\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor+2k\left(\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor+\left\lfloor\frac{dN_u(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_v(z)}{dz^2}\right\rfloor\right)\right]dz\\&+EI_{yy}\int_0^L\left[\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor-2k\left(\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor^T\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor+\left\lfloor\frac{dN_v(z)}{dz}\right\rfloor^T\left\lfloor\frac{d^2N_u(z)}{dz^2}\right\rfloor\right)\right]dz\end{aligned}\quad. \qquad (95)$$

In Appendix A, in order to obtain the element stiffness matrix in terms of the beam rigidities, length, and pretwist, the integrals in Eqn. (95) are carried out with exact integration using the shape functions derived in Section 3.2. The resulting element stiffness matrix of the pretwisted beam (see Appendix A) is as follows:

$$\left[\bar{K}^{(e)}\right]=\begin{bmatrix}\left[K_y^{(e)}\right] & \varepsilon\left[K_{yx}^{(e)}\right]\\ \varepsilon\left[K_{yx}^{(e)}\right]^T & \left[K_x^{(e)}\right]\end{bmatrix} \qquad (96)$$

where,

$$\left[K_y^{(e)}\right]=\begin{bmatrix} \frac{12EI_{xx}}{L^3} & -\frac{6EI_{xx}}{L^2} & -\frac{12EI_{xx}}{L^3} & -\frac{6EI_{xx}}{L^2} \\ & \frac{4EI_{xx}}{L} & \frac{6EI_{xx}}{L^2} & \frac{2EI_{xx}}{L} \\ & & \frac{12EI_{xx}}{L^3} & \frac{6EI_{xx}}{L^2} \\ SYM & & & \frac{4EI_{xx}}{L} \end{bmatrix} \tag{97}$$

is the submatrix associated with bending in the y-direction,

$$\left[K_x^{(e)}\right]=\begin{bmatrix} \frac{12EI_{yy}}{L^3} & \frac{6EI_{yy}}{L^2} & -\frac{12EI_{yy}}{L^3} & \frac{6EI_{yy}}{L^2} \\ & \frac{4EI_{yy}}{L} & -\frac{6EI_{yy}}{L^2} & \frac{2EI_{yy}}{L} \\ & & \frac{12EI_{yy}}{L^3} & -\frac{6EI_{yy}}{L^2} \\ SYM & & & \frac{4EI_{yy}}{L} \end{bmatrix} \tag{98}$$

is the submatrix associated with bending in the x-direction, and

$$\left[K_{yx}^{(e)}\right]=\begin{bmatrix} -\frac{6\left(EI_{xx}-EI_{yy}\right)}{L^3} & -\frac{2\left(EI_{xx}-EI_{yy}\right)}{L^2} & -\frac{6\left(EI_{xx}+EI_{yy}\right)}{L^3} & \frac{2\left(EI_{xx}+2EI_{yy}\right)}{L^2} \\ \frac{2\left(EI_{xx}-EI_{yy}\right)}{L^2} & \frac{\left(EI_{xx}-EI_{yy}\right)}{L} & \frac{2\left(2EI_{xx}+EI_{yy}\right)}{L^2} & -\frac{\left(EI_{xx}+EI_{yy}\right)}{L} \\ \frac{6\left(EI_{xx}+EI_{yy}\right)}{L^3} & \frac{2\left(EI_{xx}+2EI_{yy}\right)}{L^2} & \frac{6\left(EI_{xx}-EI_{yy}\right)}{L^3} & -\frac{2\left(EI_{xx}-EI_{yy}\right)}{L^2} \\ \frac{2\left(2EI_{xx}+EI_{yy}\right)}{L^2} & \frac{\left(EI_{xx}+EI_{yy}\right)}{L} & \frac{2\left(EI_{xx}-EI_{yy}\right)}{L^2} & -\frac{\left(EI_{xx}-EI_{yy}\right)}{L} \end{bmatrix} \tag{99}$$

is the submatrix associated with cross-coupled bending.

The submatrices associated with bending in the x- and y-directions, $[K_x^{(e)}]$ and $[K_y^{(e)}]$ respectively, are the standard stiffness matrices for planar bending. The pretwist, ε, is seen to lead to a coupling of both bending planes through submatrix $[K_{yx}^{(e)}]$. The submatrix, $[K_{yx}^{(e)}]$, is identical to that provided in [2], though the derivational details are left out of the article.

A particularly illuminating form of the element stiffness matrix is obtained by using the *primed* DOFs shown in Figure 4. The relationship between the DOFs of Section 3.2 and the primed DOFs of Figure 4 is purely geometrical:

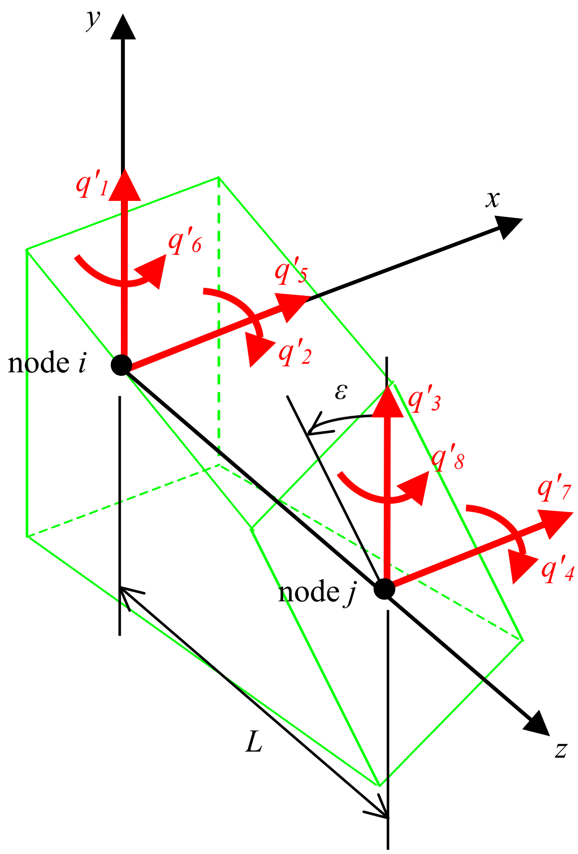


Figure 4: Pretwisted Beam Finite Element DOFs (Primed System)

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos[\varepsilon] & 0 & 0 & 0 & -\sin[\varepsilon] & 0 \\ 0 & 0 & 0 & \cos[\varepsilon] & 0 & 0 & 0 & \sin[\varepsilon] \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sin[\varepsilon] & 0 & 0 & 0 & \cos[\varepsilon] & 0 \\ 0 & 0 & 0 & -\sin[\varepsilon] & 0 & 0 & 0 & \cos[\varepsilon] \end{bmatrix} \begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \\ q'_7 \\ q'_8 \end{Bmatrix} \tag{100}$$

or,

$$\{q\} = [T']\{q'\}. \tag{101}$$

Making use of the small angle approximations ($\cos[\varepsilon]$ = 1 and $\sin[\varepsilon]$ = ε), the transformation matrix, $[T']$, relating the unprimed, $\{q\}$, and primed, $\{q'\}$, DOFs reduces to the following:

$$[T'] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{102}$$

Similar to the transformation of an element stiffness matrix from local to global coordinates, the element stiffness matrix of the pretwisted beam using the *primed* DOFs, [$\bar{K}'^{(e)}$], is obtained using the following matrix multiplication with the transformation matrix, [T'], and unprimed element stiffness matrix, [$\bar{K}^{(e)}$]:

$$\left[\bar{K}'^{(e)}\right] = [T']^T \left[\bar{K}^{(e)}\right][T'] \tag{103}$$

Performing this matrix multiplication results in the following element stiffness matrix of the pretwisted beam using the *primed* DOFs (see Appendix A):

$$\left[\bar{K}'^{(e)}\right] = \begin{bmatrix} \left[K_y^{(e)}\right] & \varepsilon\left[K'^{(e)}_{yx}\right] \\ \varepsilon\left[K'^{(e)}_{yx}\right]^T & \left[K_x^{(e)}\right] \end{bmatrix} \tag{104}$$

where the main-diagonal submatrices are identical to those of the unprimed element stiffness matrix and the *primed* cross-coupling submatrix, [$K'^{(e)}_{yx}$], is as follows:

$$\left[K'^{(e)}_{yx}\right] = \begin{bmatrix} -\frac{6(EI_{xx}-EI_{yy})}{L^3} & -\frac{2(EI_{xx}-EI_{yy})}{L^2} & \frac{6(EI_{xx}-EI_{yy})}{L^3} & -\frac{4(EI_{xx}-EI_{yy})}{L^2} \\ \frac{2(EI_{xx}-EI_{yy})}{L^2} & \frac{(EI_{xx}-EI_{yy})}{L} & -\frac{2(EI_{xx}-EI_{yy})}{L^2} & \frac{(EI_{xx}-EI_{yy})}{L} \\ \frac{6(EI_{xx}-EI_{yy})}{L^3} & \frac{2(EI_{xx}-EI_{yy})}{L^2} & -\frac{6(EI_{xx}-EI_{yy})}{L^3} & \frac{4(EI_{xx}-EI_{yy})}{L^2} \\ \frac{4(EI_{xx}-EI_{yy})}{L^2} & \frac{(EI_{xx}-EI_{yy})}{L} & -\frac{4(EI_{xx}-EI_{yy})}{L^2} & \frac{3(EI_{xx}-EI_{yy})}{L} \end{bmatrix}. \tag{105}$$

In this form, it is evident that there is no cross coupling between bending planes if (1) the pretwist, ε, is zero *or* (2), the rigidities in both directions are identical (i.e., the cross section is circular so that $EI_{xx} = EI_{yy}$).

Appendix A also contains a rank test on the primed element stiffness matrix, [$\bar{K}'^{(e)}$]. As expected, the number of zero-valued eigenvalues is four, which is the correct number of rigid body modes for this element—that is, two translational and two rotational rigid body modes.

4 FEM Analysis Program for Cantilevered, Pretwisted Beams

As discussed in Section 2.1, any form of pretwist can be achieved by axially connecting a sufficiently large number of pretwisted beam finite elements together. Making use of the element stiffness matrix derived in Section 3, this Section presents the development of an FEM analysis program for *cantilevered* beams that are initially straight, but pretwisted, prior to loading.

4.1 Globalization

When interconnecting multiple pretwisted beam elements together during the assembly process, it is easiest to work in global coordinates. An illustration of the eight global DOFs of a single pretwisted beam element is provided in Figure 5. DOFs u_{Xi} and u_{Yi} correspond to the translational deflections of node i in the global X- and Y-directions respectively. DOFs θ_{Xi} and θ_{Yi} correspond to the rotations of node i about the global X- and Y-axes respectively. The four DOFs for node j are similar. The transformation relating the *primed* local element DOFs to these global DOFs is given by:

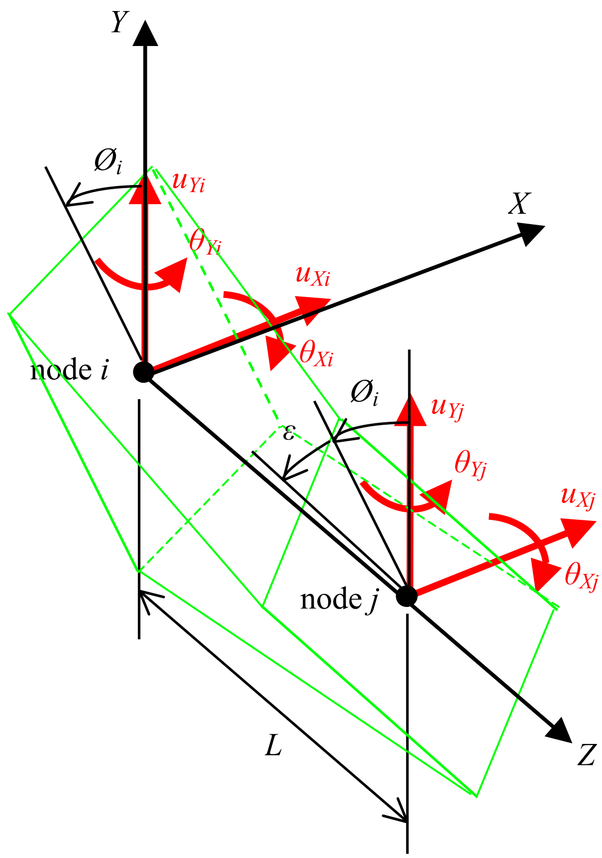


Figure 5: Pretwisted Beam Finite Element DOFs (Global System)

$$\begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \\ q'_7 \\ q'_8 \end{Bmatrix} = \begin{bmatrix} -\sin[\phi_i] & 0 & \cos[\phi_i] & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos[\phi_i] & 0 & \sin[\phi_i] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin[\phi_i] & 0 & \cos[\phi_i] & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos[\phi_i] & 0 & \sin[\phi_i] \\ \cos[\phi_i] & 0 & \sin[\phi_i] & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin[\phi_i] & 0 & \cos[\phi_i] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos[\phi_i] & 0 & \sin[\phi_i] & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin[\phi_i] & 0 & \cos[\phi_i] \end{bmatrix} \begin{Bmatrix} u_{Xi} \\ \theta_{Xi} \\ u_{Yi} \\ \theta_{Yi} \\ u_{Xj} \\ \theta_{Xj} \\ u_{Yj} \\ \theta_{Yj} \end{Bmatrix} \tag{106}$$

where $\varnothing_i$ is the orientation of the principle axes x,y for the centroid at node i relative to the global X,Y coordinate system. The square transformation matrix in Eqn. (106) is labeled $[T^{(e)}]$ in the subsequent analysis.

As in the case of simpler finite elements, the force transformation matrix relating the global nodal forces to the local nodal forces is given by the transpose of $[T^{(e)}]$. Also, the transformation equation relating the primed element stiffness matrix to the global element stiffness matrix, $[K^{(e)}]$, is given by:

$$\left[K^{(e)}\right] = \left[T^{(e)}\right]^T \left[\bar{K}'^{(e)}\right]\left[T^{(e)}\right]. \tag{107}$$

This global element stiffness matrix relates the eight global nodal translations and rotations, u_{Xi} through θ_{Yj}, to the eight global nodal forces and moments F_{Xi} through M_{Yj}. The global nodal forces, F, and moments, M, follow respectively the same sign conventions as the nodal translations and rotations as shown in Figure 5.

4.2 Assembly of the Master Stiffness Matrix and Forcing Vector

Since the structure of each global element stiffness matrix per Eqn. (107) is identical to that of the finite element formulation for plane beam bending, the assembly process is also identical. Upon merging all of the global element stiffness matrices together, the master stiffness matrix for pretwisted beam bending is formed.

The application of nodal loading in the pretwisted beam finite element model is also identical to that of the process used to apply nodal loads to a plane beam finite element model.

An FEM program for analyzing pretwisted beams, as written in *Mathematica*, is provided in Appendix B. The first cell contains Module "PretwistedBeamElementStiffnessMatrixLocal", which computes the primed element stiffness matrix derived in Section 3. The second cell containing Module "PretwistedBeamElementStiffnessMatrixGlobal" computes the global element stiffness matrix per Eqn. (107). The third cell, containing Module "PretwistedBeamMasterStiffness", assembles the master stiffness matrix in global coordinates so that the global displacement vector is organized as follows:

$$\{u\} = \lfloor u_{X1} \quad \theta_{X1} \quad u_{Y1} \quad \theta_{Y1} \quad u_{X2} \quad \theta_{X2} \quad u_{Y2} \quad \theta_{Y2} \quad \ldots \quad u_{XN} \quad \theta_{XN} \quad u_{YN} \quad \theta_{YN} \rfloor^T \tag{108}$$

where the nodes are numbered with node 1 at the cantilevered end, increasing with Z, up to node N at the free end—the total number of nodes being N.

4.3 Application of Boundary Conditions and Solution

Since node 1 represents the cantilevered end of the pretwisted beam, all four of its freedoms must be fixed at zero. Note that four fixed nodal freedoms is the minimum number necessary to remove all of the free-free beam's rigid body modes. This is done in cell four, Module "CantileveredPretwistedBeamSolution", of the *Mathematica* program provided in Appendix B, by zeroing out the first four rows and columns of the master stiffness matrix and the first four forcing terms and by specifying ones on the first four main diagonal elements of the modified master stiffness matrix. Module "CantileveredPretwistedBeamSolution" also contains a solution of the modified system the global displacements through use of *Mathematica*'s built in matrix inverse solver. The nodal forces, including reactions at the cantilevered end, are recovered by premultiplying the global displacement vector solution by the original master stiffness matrix.

5 Results and Discussion

This Section applies the analysis program discussed in Section 4 to two problems associated with static deflections of cantilevered beams that are initially straight, but pretwisted, prior to loading.

5.1 Analytical Results

Due to the computational power necessary to solve a problem analytically, where all properties of the solution are left in variable form, analytical solutions are limited to cantilevered beam problems composed of very few elements. However, solutions to analytical problems are insightful as they offer an explanation of the general phenomenon associated with bending of pretwisted, cantilevered beams. This Section is concerned with the analytical tip deflection solution of a cantilevered beam of length, L, with a small overall pretwist, ε, which is loaded by a single point force, P, at the tip of the beam. Since the beam is loaded only at the tip and since the overall pretwist is assumed to be small, the problem can be solved *exactly* using only a single pretwisted beam finite element. The problem is illustrated in Figure 6. As shown, the tip force is fixed in space and applied in the negative Y-direction.

Using the *Mathematica* FEM analysis program for cantilevered, pretwisted beams supplied in Appendix B and documented in Section 4, along with a simple driver program supplied in Appendix C, the analytical solution to the problem illustrated in Figure 6, neglecting all products of ε, is easily found to be:

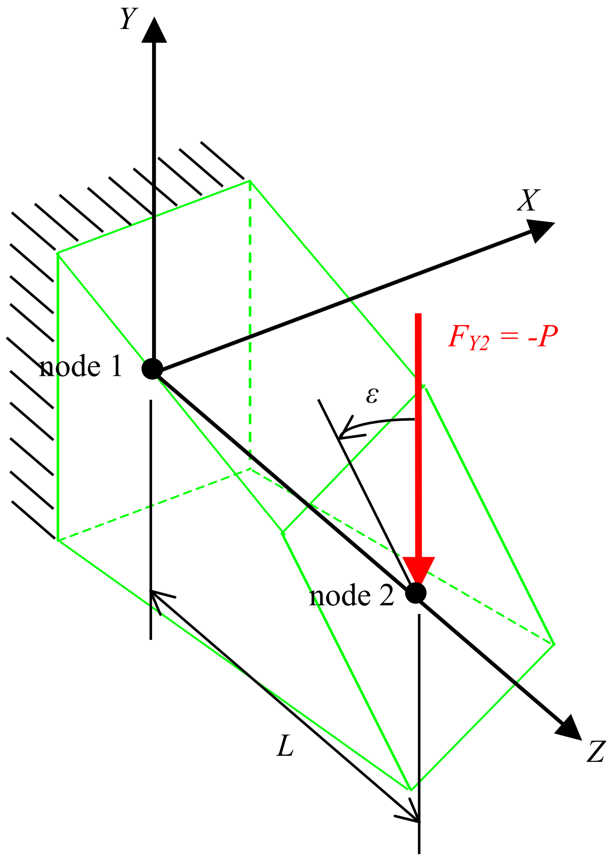


Figure 6: Bending of a Pretwisted, Cantilevered Beam with Small Overall Pretwist

$$\begin{Bmatrix} u_{X1} \\ \theta_{X1} \\ u_{Y1} \\ \theta_{Y1} \\ u_{X2} \\ \theta_{X2} \\ u_{Y2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{\varepsilon}{4}\left(\dfrac{EI_{xx}}{EI_{yy}}-1\right)\left(\dfrac{PL^3}{3EI_{xx}}\right) \\ \dfrac{PL^2}{2EI_{xx}} \\ -\dfrac{PL^3}{3EI_{xx}} \\ -\dfrac{\varepsilon}{3}\left(\dfrac{EI_{xx}}{EI_{yy}}-1\right)\left(\dfrac{PL^2}{2EI_{xx}}\right) \end{Bmatrix} \tag{109}$$

and

$$\begin{Bmatrix} F_{X1} \\ M_{X1} \\ F_{Y1} \\ M_{Y1} \\ F_{X2} \\ M_{X2} \\ F_{Y2} \\ M_{Y2} \end{Bmatrix}_{recovered} = \begin{Bmatrix} 0 \\ -PL \\ P \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \end{Bmatrix}. \tag{110}$$

The root reaction forces and moments of node 1 are the exact reaction loads resulting from a static analysis of the beam.

The Y-direction tip deflection, u_{Y2}, and X-direction tip slope, θ_{X2}, are identical to the Bernoulli-Euler solution of an unpretwisted, cantilevered beam loaded by tip force P in the negative Y-direction, as given by any good Mechanics of Materials book. Unlike the classical Bernoulli-Euler solution of an unpretwisted, cantilevered beams however, the pretwist is seen to bring about a deflection and corresponding slope normal to the applied load. The deflection normal to the applied force is a scaled version of the primary vertical deflection and depends on the ratio of EI_{xx} to EI_{yy} as follows:

$$u_{X2} = \frac{\varepsilon}{4}\left(\frac{EI_{xx}}{EI_{yy}} - 1\right)u_{Y2}. \tag{111}$$

This is identical to the solution provided in [2]. For deep-webbed beams loaded along the web, the ratio of EI_{xx} to EI_{yy} can be quite large. Then, even a small overall pretwist can lead to a substantial deflection normal to the applied load.

If either the pretwist is zero, or both rigidities are identical (i.e., $EI_{xx} = EI_{yy}$), such as for the case of a prismatic beam with circular cross section, it seen that the deflection and corresponding slope normal to the applied load vanish. This is consistent with physical intuition.

5.2 Benchmark Problem

A common benchmark problem to determine the effects of warping in shell and solid finite elements is that of a tip-loaded, cantilevered beam, with 90˚ of overall pretwist. Naturally, the same benchmark problem is well suited for testing the pretwisted beam finite element developed in this paper. The problem is illustrated in Figure 7. For a beam of length, L = 12, Modulus of Elasticity, E = 29 x 10^6, Poisson's ratio, v = 0.22, web width, w = 1.1, web height, h = 0.32, and an overall pretwist of 90˚, the theoretical solutions of tip deflections in the direction of loading are 0.005424 and 0.001754 for independent tip loading of one unit in the X- and Y-directions respectively as given by [3].

The driver program supplied in Appendix D is used to solve this problem numerically using a number of equally-spaced pretwisted finite elements. Solutions are obtained using 5 through 40 equally-spaced pretwisted finite elements in steps of 5. The results are provided in Table 1 below:

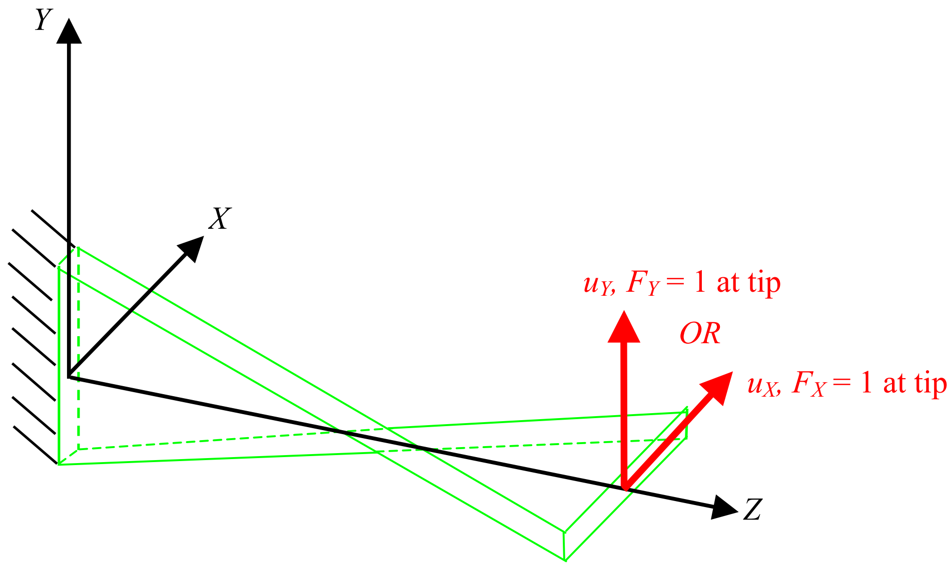


Figure 7: Bending of a Pretwisted, Cantilevered Beam with 90˚ Overall Pretwist

Table 1: Numerical Solutions to the 90˚ Pretwist Benchmark Problem

Number of Elements	X-Dir. Tip Displacement	Y-Dir. Tip Displacement	Normalized X-Dir. Tip Displacement	Normalized Y-Dir. Tip Displacement
5	0.00918174	0.00277751	1.692798673	1.583529076
10	0.00592939	0.00187257	1.093176622	1.067599772
15	0.00563719	0.00179809	1.039304941	1.02513683
20	0.00554308	0.00177469	1.021954277	1.011795895
25	0.00550056	0.00176425	1.014115044	1.005843786
30	0.00547768	0.00175867	1.009896755	1.002662486
35	0.00546393	0.00175533	1.007361726	1.000758267
40	0.00545502	0.00175318	1.005719027	0.999532497

The normalized tip displacements correspond to the computed tip displacements divided by the tip displacements from the theoretical solutions. Typical engineering accuracy, which is approximately 2% error, is obtained using 15 to 20 elements. As the number of elements increases to 35 or 40, the numerical finite element solution tends toward the exact solution. The derived pretwisted finite element appears to be quite accurate.

6 Conclusions

Research associated with rotor blades built with pretwisted spars and research associated with warpage in prismatic beams motivates the study of pretwisted beams. In a pretwisted beam, the principle axes of a cross section rotate along the beam's length. By ignoring axial deflection, torsion, and the effects of shear deformation, this paper derives an eight DOF pretwisted beam element from first principles. In order to derive the element, it must be assumed that the overall pretwist across a single element be small, such that small angle approximations apply. This is not as restrictive as it first may sound, since any form of pretwist can be achieved by axially connecting a sufficiently large number of elements together.

The pretwisted beam element is tested for cases of static deflection of a cantilevered beam. As shown, the pretwist leads to a coupling of bending in both planes. Also, like in unsymmetric bending, deflections of a cantilevered, pretwisted beam exhibit components both parallel and normal to the direction of loading. The deflection normal to the direction of loading can be substantial if the beam is deep-webbed, even if the overall pretwist is small. Comparison of finite element analysis results for the benchmark case of a beam with 90˚ of overall pretwist show excellent convergence to the theoretical solution, indicating that the pretwisted beam element, as developed, is quite accurate.

If either the pretwist is zero or both rigidities are identical (i.e., $EI_{xx} = EI_{yy}$), such as for the case of a prismatic beam with circular cross section, the deflection normal to the applied load vanishes and the results agree exactly with results obtained using classical Bernoulli-Euler beam theory.

7 References

[1] Banarjee, J. R., "Free Vibration Analysis of a Twisted Beam Using the Dynamic Stiffness Method", International Journal of Solids and Structures, vol. 38, pp. 6703-6722, 2001.

[2] Vielsack, P., "Lateral Bending Vibrations of a Beam with Small Pretwist", Engineering Structures, vol. 22, pp. 691-698, 2000.

[3] MacNeal, R. H. and Harder, R. L., "A Proposed Standard Set of Problems to Test Finite Element Accuracy", Finite Elements in Analysis and Design, vol. 1, pp. 3-20, 1985.

Appendix A: Derivation of the Shape Functions and Element Stiffness Matrix Using *Mathematica*

This script derives the shape functions and element stiffness matrix of a pretwisted beam element:

```
ClearAll[ L, EIxx, EIyy, ξ, ∈ ];


(* Construct the constraint matrix relating end conditions to unknown
   polynomial coefficients: *)

Constraint = { { 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 },
            { 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 },
            { 1, 1, 1, 1, 1, 0, 0, 0, 0, 0 },
            { 0, 0, 0, 0, 0, 1, 1, 1, 1, 1 },
            { -∈, 0, 0, 0, 0, 0, -1, 0, 0, 0 },
            { 0, 1, 0, 0, 0, -∈, 0, 0, 0, 0 },
            { -∈, -∈, -∈, -∈, -∈, 0, -1, -2, -3, -4 },
            { 0, 1, 2, 3, 4, -∈, -∈, -∈, -∈, -∈ },
            { 0, 0, 0, 0, 2*EIyy, 0, 0, 0, -∈*( EIyy + EIxx ),
             -4*∈*ξ*( EIyy + EIxx ) },
            { 0, 0, 0, ∈*( EIyy + EIxx ), 4*∈*ξ*( EIyy + EIxx ), 0, 0, 0, 0,
             2*EIxx } };


(* Calculate its inverse, ignoring all terms involving products of k: *)

ConstraintInv = Simplify[ Inverse[ Constraint ] ];
ConstraintInv = Simplify[ ConstraintInv /. { ∈^2 -> 0, ∈^3 -> 0 } ];


(* Calculate the unknown polynomial coefficients of the shape functions by
   setting each DOF to unity in turn: *)

{ { a01, a02, a03, a04, a05, a06, a07, a08 },
  { a11, a12, a13, a14, a15, a16, a17, a18 },
  { a21, a22, a23, a24, a25, a26, a27, a28 },
  { a31, a32, a33, a34, a35, a36, a37, a38 },
  { a41, a42, a43, a44, a45, a46, a47, a48 },
  { b01, b02, b03, b04, b05, b06, b07, b08 },
  { b11, b12, b13, b14, b15, b16, b17, b18 },
  { b21, b22, b23, b24, b25, b26, b27, b28 },
  { b31, b32, b33, b34, b35, b36, b37, b38 },
  { b41, b42, b43, b44, b45, b46, b47, b48 }  } =
                                   Simplify[ ConstraintInv.{ { 0, 0, 0, 0, 1, 0, 0, 0 },
```

```
                                   { 1, 0, 0, 0, 0, 0, 0, 0 },
                                   { 0, 0, 0, 0, 0, 0, 1, 0 },
                                   { 0, 0, 1, 0, 0, 0, 0, 0 },
                                   { 0, L, 0, 0, 0, 0, 0, 0 },
                                   { 0, 0, 0, 0, 0, L, 0, 0 },
                                   { 0, 0, 0, L, 0, 0, 0, 0 },
                                   { 0, 0, 0, 0, 0, 0, 0, L },
                                   { 0, 0, 0, 0, 0, 0, 0, 0 },
                                   { 0, 0, 0, 0, 0, 0, 0, 0 }  } ];


(* Assemble the shape functions and their 1st and 2nd derivatives: *)

Nu1 [ξ_] = a01 + a11*ξ + a21*ξ^2 + a31*ξ^3 + a41*ξ^4;
Nv1 [ξ_] = b01 + b11*ξ + b21*ξ^2 + b31*ξ^3 + b41*ξ^4;
Nu2 [ξ_] = a02 + a12*ξ + a22*ξ^2 + a32*ξ^3 + a42*ξ^4;
Nv2 [ξ_] = b02 + b12*ξ + b22*ξ^2 + b32*ξ^3 + b42*ξ^4;
Nu3 [ξ_] = a03 + a13*ξ + a23*ξ^2 + a33*ξ^3 + a43*ξ^4;
Nv3 [ξ_] = b03 + b13*ξ + b23*ξ^2 + b33*ξ^3 + b43*ξ^4;
Nu4 [ξ_] = a04 + a14*ξ + a24*ξ^2 + a34*ξ^3 + a44*ξ^4;
Nv4 [ξ_] = b04 + b14*ξ + b24*ξ^2 + b34*ξ^3 + b44*ξ^4;
Nu5 [ξ_] = a05 + a15*ξ + a25*ξ^2 + a35*ξ^3 + a45*ξ^4;
Nv5 [ξ_] = b05 + b15*ξ + b25*ξ^2 + b35*ξ^3 + b45*ξ^4;
Nu6 [ξ_] = a06 + a16*ξ + a26*ξ^2 + a36*ξ^3 + a46*ξ^4;
Nv6 [ξ_] = b06 + b16*ξ + b26*ξ^2 + b36*ξ^3 + b46*ξ^4;
Nu7 [ξ_] = a07 + a17*ξ + a27*ξ^2 + a37*ξ^3 + a47*ξ^4;
Nv7 [ξ_] = b07 + b17*ξ + b27*ξ^2 + b37*ξ^3 + b47*ξ^4;
Nu8 [ξ_] = a08 + a18*ξ + a28*ξ^2 + a38*ξ^3 + a48*ξ^4;
Nv8 [ξ_] = b08 + b18*ξ + b28*ξ^2 + b38*ξ^3 + b48*ξ^4;

Nu  [ξ_] = { { Nu1[ξ], Nu2[ξ], Nu3[ξ], Nu4[ξ], Nu5[ξ], Nu6[ξ], Nu7[ξ], Nu8[ξ] } };
Nv  [ξ_] = { { Nv1[ξ], Nv2[ξ], Nv3[ξ], Nv4[ξ], Nv5[ξ], Nv6[ξ], Nv7[ξ], Nv8[ξ] } };

NuD1[ξ_] = D[ Nu[ξ], {ξ,1} ]/L;
NvD1[ξ_] = D[ Nv[ξ], {ξ,1} ]/L;

NuD2[ξ_] = D[ Nu[ξ], {ξ,2} ]/(L^2);
NvD2[ξ_] = D[ Nv[ξ], {ξ,2} ]/(L^2);


(* Crosscheck for completeness: *)

SumNu[ξ_] = Nu1[ξ] + Nu2[ξ] + Nu3[ξ] + Nu4[ξ] + Nu5[ξ] + Nu6[ξ] + Nu7[ξ] + Nu8[ξ];
SumNv[ξ_] = Nv1[ξ] + Nv2[ξ] + Nv3[ξ] + Nv4[ξ] + Nv5[ξ] + Nv6[ξ] + Nv7[ξ] + Nv8[ξ];
```

```
(* Calculate the element stiffness matrix, ignoring all terms involving products
   of k: *)

KeBar = Simplify[ L*EIyy*Integrate[        Transpose[NuD2[ξ]].NuD2[ξ]
                                   -2*∈/L*( Transpose[NuD2[ξ]].NvD1[ξ]
                                            +Transpose[NvD1[ξ]].NuD2[ξ]),{ξ,0,1}]
                  +L*EIxx*Integrate[        Transpose[NvD2[ξ]].NvD2[ξ]
                                   +2*∈/L*( Transpose[NvD2[ξ]].NuD1[ξ]
                                            +Transpose[NuD1[ξ]].NvD2[ξ]),{ξ,0,1}]
            ];
KeBar = Simplify[ KeBar /. { ∈^2 -> 0, ∈^3 -> 0 } ];


(* Calculate the element stiffness matrix in the primed system, ignoring all
   terms involving products of k: *)

Te = { { 1, 0, 0,  0, 0, 0,  0, 0 },
       { 0, 1, 0,  0, 0, 0,  0, 0 },
       { 0, 0, 1,  0, 0, 0, -∈, 0 },
       { 0, 0, 0,  1, 0, 0,  0, ∈ },
       { 0, 0, 0,  0, 1, 0,  0, 0 },
       { 0, 0, 0,  0, 0, 1,  0, 0 },
       { 0, 0, ∈,  0, 0, 0,  1, 0 },
       { 0, 0, 0, -∈, 0, 0,  0, 1 }  };

KeBarPrime = Simplify[ Transpose[ Te ].KeBar.Te ];
KeBarPrime = Simplify[ KeBarPrime /. { ∈^2 -> 0, ∈^3 -> 0 } ];


(* Perform a rank test on KeBarPrime: *)

Lamda = Eigenvalues[ KeBarPrime ];
Lamda = Lamda /. { ∈^2 -> 0, ∈^3 -> 0, ∈^4 -> 0 };


(* Print some of the results: *)

Print[ "Constraint Matrix = ", Constraint // MatrixForm ];
Print[ "Constraint Matrix Inverse = ", ConstraintInv // MatrixForm ];

Print[ "Nv1 = ", Nv1[ξ] ];
Print[ "Nu1 = ", Nu1[ξ] ];
Print[ "Nv2 = ", Nv2[ξ] ];
Print[ "Nu2 = ", Nu2[ξ] ];
Print[ "Nv3 = ", Nv3[ξ] ];
Print[ "Nu3 = ", Nu3[ξ] ];
```

```
Print[ "Nv4 = ", Nv4[ξ] ];
Print[ "Nu4 = ", Nu4[ξ] ];
Print[ "Nv5 = ", Nv5[ξ] ];
Print[ "Nu5 = ", Nu5[ξ] ];
Print[ "Nv6 = ", Nv6[ξ] ];
Print[ "Nu6 = ", Nu6[ξ] ];
Print[ "Nv7 = ", Nv7[ξ] ];
Print[ "Nu7 = ", Nu7[ξ] ];
Print[ "Nv8 = ", Nv8[ξ] ];
Print[ "Nu8 = ", Nu8[ξ] ];

Print[ "SumNv = ", Simplify[ SumNv[1/2] ] ];
Print[ "SumNu = ", Simplify[ SumNu[1/2] ] ];

Print[ "KeBar = ", KeBar // MatrixForm ];

Print[ "KeBarPrime = ", KeBarPrime // MatrixForm ];
(*Print[ "KeBarPrime = ", KeBarPrime // InputForm ];*)

Print[ "Eigenvalue 1 = ", Lamda[[1]] ];
Print[ "Eigenvalue 2 = ", Lamda[[2]] ];
Print[ "Eigenvalue 3 = ", Lamda[[3]] ];
Print[ "Eigenvalue 4 = ", Lamda[[4]] ];
(*Print[ "Eigenvalue 5 = ", Lamda[[5]] ];
Print[ "Eigenvalue 6 = ", Lamda[[6]] ];
Print[ "Eigenvalue 7 = ", Lamda[[7]] ];
Print[ "Eigenvalue 8 = ", Lamda[[8]] ];*)
```

$$\text{Constraint Matrix} = \left(\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
-\epsilon & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \\
-\epsilon & -\epsilon & -\epsilon & -\epsilon & -\epsilon & 0 & -1 & -2 & -3 & -4 \\
0 & 1 & 2 & 3 & 4 & -\epsilon & -\epsilon & -\epsilon & -\epsilon & -\epsilon \\
0 & 0 & 0 & 0 & 2\,EIyy & 0 & 0 & 0 & -(EIxx+EIyy)\,\epsilon & -4\,(EIxx+EIyy)\,\epsilon\,\xi \\
0 & 0 & 0 & (EIxx+EIyy)\,\epsilon & 4\,(EIxx+EIyy)\,\epsilon\,\xi & 0 & 0 & 0 & 0 & 2\,EIxx
\end{array}\right)$$

$$\text{Constraint Matrix Inverse} = \left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0 & 0 & 1 & 0 & 0 \\
-3 & \frac{(EIxx-EIyy)\,\epsilon}{EIyy} & 3 & -\frac{(EIxx+2\,EIyy)\,\epsilon}{EIyy} & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIyy} & -2 & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIyy} & -1 \\
2 & -\frac{(2\,EIxx+EIyy)\,\epsilon}{EIyy} & -2 & \left(3+\frac{2\,EIxx}{EIyy}\right)\epsilon & \frac{(EIxx+EIyy)\,\epsilon}{EIyy} & 1 & \frac{(EIxx+EIyy)\,\epsilon}{EIyy} & 1 \\
0 & \frac{(EIxx+EIyy)\,\epsilon}{EIyy} & 0 & -\frac{(EIxx+EIyy)\,\epsilon}{EIyy} & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIyy} & 0 & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIyy} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\epsilon & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\epsilon-\frac{EIyy\,\epsilon}{EIxx} & -3 & \frac{(2\,EIxx+EIyy)\,\epsilon}{EIxx} & 3 & 2 & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIxx} & 1 & -\frac{(EIxx+EIy}{2\,EIxx} \\
\epsilon+\frac{2\,EIyy\,\epsilon}{EIxx} & 2 & \left(-3-\frac{2\,EIyy}{EIxx}\right)\epsilon & -2 & -1 & \frac{(EIxx+EIyy)\,\epsilon}{EIxx} & -1 & \frac{(EIxx+EIyy}{EIxx} \\
-\frac{(EIxx+EIyy)\,\epsilon}{EIxx} & 0 & \frac{(EIxx+EIyy)\,\epsilon}{EIxx} & 0 & 0 & -\frac{(EIxx+EIyy)\,\epsilon}{2\,EIxx} & 0 & -\frac{(EIxx+EIy}{2\,EIxx}
\end{array}\right.$$

$$Nv1 = 1 - 3\,\xi^2 + 2\,\xi^3$$

$$Nu1 = \epsilon\,\xi + \frac{(EIxx - EIyy)\,\epsilon\,\xi^2}{EIyy} - \frac{(2\,EIxx + EIyy)\,\epsilon\,\xi^3}{EIyy} + \frac{(EIxx + EIyy)\,\epsilon\,\xi^4}{EIyy}$$

$$Nv2 = -L\,\xi + 2\,L\,\xi^2 - L\,\xi^3$$

$$Nu2 = -\frac{(EIxx + EIyy)\,L\,\epsilon\,\xi^2}{2\,EIyy} + \frac{(EIxx + EIyy)\,L\,\epsilon\,\xi^3}{EIyy} - \frac{(EIxx + EIyy)\,L\,\epsilon\,\xi^4}{2\,EIyy}$$

$$\mathrm{Nv3} = 3\,\xi^2 - 2\,\xi^3$$

$$\mathrm{Nu3} = -\frac{(\mathrm{EIxx} + 2\,\mathrm{EIyy})\,\epsilon\,\xi^2}{\mathrm{EIyy}} + \left(3 + \frac{2\,\mathrm{EIxx}}{\mathrm{EIyy}}\right)\epsilon\,\xi^3 - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\epsilon\,\xi^4}{\mathrm{EIyy}}$$

$$\mathrm{Nv4} = \mathrm{L}\,\xi^2 - \mathrm{L}\,\xi^3$$

$$\mathrm{Nu4} = -\frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^2}{2\,\mathrm{EIyy}} + \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^3}{\mathrm{EIyy}} - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^4}{2\,\mathrm{EIyy}}$$

$$\mathrm{Nv5} = -\epsilon\,\xi + \left(\epsilon - \frac{\mathrm{EIyy}\,\epsilon}{\mathrm{EIxx}}\right)\xi^2 + \left(\epsilon + \frac{2\,\mathrm{EIyy}\,\epsilon}{\mathrm{EIxx}}\right)\xi^3 - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\epsilon\,\xi^4}{\mathrm{EIxx}}$$

$$\mathrm{Nu5} = 1 - 3\,\xi^2 + 2\,\xi^3$$

$$\mathrm{Nv6} = -\frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^2}{2\,\mathrm{EIxx}} + \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^3}{\mathrm{EIxx}} - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^4}{2\,\mathrm{EIxx}}$$

$$\mathrm{Nu6} = \mathrm{L}\,\xi - 2\,\mathrm{L}\,\xi^2 + \mathrm{L}\,\xi^3$$

$$\mathrm{Nv7} = \frac{(2\,\mathrm{EIxx} + \mathrm{EIyy})\,\epsilon\,\xi^2}{\mathrm{EIxx}} + \left(-3 - \frac{2\,\mathrm{EIyy}}{\mathrm{EIxx}}\right)\epsilon\,\xi^3 + \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\epsilon\,\xi^4}{\mathrm{EIxx}}$$

$$\mathrm{Nu7} = 3\,\xi^2 - 2\,\xi^3$$

$$\mathrm{Nv8} = -\frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^2}{2\,\mathrm{EIxx}} + \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^3}{\mathrm{EIxx}} - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon\,\xi^4}{2\,\mathrm{EIxx}}$$

$$\mathrm{Nu8} = -\mathrm{L}\,\xi^2 + \mathrm{L}\,\xi^3$$

$$\mathrm{SumNv} = 1 - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon}{16\,\mathrm{EIxx}}$$

$$\mathrm{SumNu} = 1 - \frac{(\mathrm{EIxx} + \mathrm{EIyy})\,\mathrm{L}\,\epsilon}{16\,\mathrm{EIyy}}$$

$$\mathrm{KeBar} = \begin{pmatrix}
\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & -\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & -\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{6\,(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}+2\,\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} \\
-\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIxx}}{\mathrm{L}} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIxx}}{\mathrm{L}} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{2\,(2\,\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} \\
-\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}+2\,\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & -\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} \\
-\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIxx}}{\mathrm{L}} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIxx}}{\mathrm{L}} & \frac{2\,(2\,\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} \\
-\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(2\,\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} \\
-\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{2\,(\mathrm{EIxx}+2\,\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIyy}}{\mathrm{L}} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIyy}}{\mathrm{L}} \\
-\frac{6\,(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(2\,\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} \\
\frac{2\,(\mathrm{EIxx}+2\,\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{(\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & -\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIyy}}{\mathrm{L}} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIyy}}{\mathrm{L}}
\end{pmatrix}$$

$$\mathrm{KeBarPrime} = \begin{pmatrix}
\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & -\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & -\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{4\,(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} \\
-\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIxx}}{\mathrm{L}} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIxx}}{\mathrm{L}} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{2\,(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} \\
-\frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{12\,\mathrm{EIxx}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} \\
-\frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIxx}}{\mathrm{L}} & \frac{6\,\mathrm{EIxx}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIxx}}{\mathrm{L}} & \frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & -\frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{3\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} \\
-\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} \\
-\frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{2\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIyy}}{\mathrm{L}} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIyy}}{\mathrm{L}} \\
\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & \frac{2\,(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{6\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^3} & -\frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & -\frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{12\,\mathrm{EIyy}}{\mathrm{L}^3} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} \\
\frac{4\,(-\mathrm{EIxx}+\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{4\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}^2} & \frac{3\,(\mathrm{EIxx}-\mathrm{EIyy})\,\epsilon}{\mathrm{L}} & \frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{2\,\mathrm{EIyy}}{\mathrm{L}} & -\frac{6\,\mathrm{EIyy}}{\mathrm{L}^2} & \frac{4\,\mathrm{EIyy}}{\mathrm{L}}
\end{pmatrix}$$

Eigenvalue 1 = 0

Eigenvalue 2 = 0

Eigenvalue 3 = 0

Eigenvalue 4 = 0

Appendix B: FEM Analysis Program for Cantilevered, Pretwisted Beams Using *Mathematica*

This Module implements the element stiffness matrix of a pretwisted beam element in local coordinates. That is, the principle axes at node i are untwisted (reference) and the principle axes at node j are pretwisted by an amount "∈" about Z relative to that of i. All DOFs are along or about the principle axes of node 1. The implemented stiffness matrix assumes ∈ << 1 radian. The Z-axis is directed from node i to node j, which are separated by a distance "L".

```
PretwistedBeamElementStiffnessLocal[ L_, ∈_, EIxx_, EIyy_, numer_ ] :=
        Module[
                { KeBar },

                KeBar =  {{(12*EIxx)/L^3, (-6*EIxx)/L^2, (-12*EIxx)/L^3, (-6*EIxx)/L^2,
                           (-6*(EIxx - EIyy)*∈)/L^3, (-2*(EIxx - EIyy)*∈)/L^2,
                           (6*(EIxx - EIyy)*∈)/L^3, (4*(-EIxx + EIyy)*∈)/L^2         },
                          {(-6*EIxx)/L^2, (4*EIxx)/L, (6*EIxx)/L^2, (2*EIxx)/L,
                           (2*(EIxx - EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L,
                           (2*(-EIxx + EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L            },
                          {(-12*EIxx)/L^3, (6*EIxx)/L^2, (12*EIxx)/L^3, (6*EIxx)/L^2,
                           (6*(EIxx - EIyy)*∈)/L^3, (2*(EIxx - EIyy)*∈)/L^2,
                           (-6*(EIxx - EIyy)*∈)/L^3, (4*(EIxx - EIyy)*∈)/L^2         },
                          {(-6*EIxx)/L^2, (2*EIxx)/L, (6*EIxx)/L^2, (4*EIxx)/L,
                           (4*(EIxx - EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L,
                           (-4*(EIxx - EIyy)*∈)/L^2, (3*(EIxx - EIyy)*∈)/L           },
                          {(-6*(EIxx - EIyy)*∈)/L^3, (2*(EIxx - EIyy)*∈)/L^2,
                           (6*(EIxx - EIyy)*∈)/L^3, (4*(EIxx - EIyy)*∈)/L^2,
                           (12*EIyy)/L^3, (6*EIyy)/L^2, (-12*EIyy)/L^3, (6*EIyy)/L^2   },
                          {(-2*(EIxx - EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L,
                           (2*(EIxx - EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L,
                           (6*EIyy)/L^2, (4*EIyy)/L, (-6*EIyy)/L^2, (2*EIyy)/L        },
                          {(6*(EIxx - EIyy)*∈)/L^3, (2*(-EIxx + EIyy)*∈)/L^2,
                           (-6*(EIxx - EIyy)*∈)/L^3, (-4*(EIxx - EIyy)*∈)/L^2,
                           (-12*EIyy)/L^3, (-6*EIyy)/L^2, (12*EIyy)/L^3, (-6*EIyy)/L^2 },
                          {(4*(-EIxx + EIyy)*∈)/L^2, ((EIxx - EIyy)*∈)/L,
                           (4*(EIxx - EIyy)*∈)/L^2, (3*(EIxx - EIyy)*∈)/L,
                           (6*EIyy)/L^2, (2*EIyy)/L, (-6*EIyy)/L^2, (4*EIyy)/L        } };

                If[ numer, Return[ N[ KeBar ] ], Return[ KeBar ] ];
        ];
```

This Module implements the element stiffness matrix of a pretwisted beam element in global X/Y/Z coordinates. The principle axes at node i are pretwisted by an angle "ϕi" about Z and the principle axes at node j are pretwisted by an amount "ϕj" about Z. The implemented stiffness matrix assumes ϕj - ϕi = ∈ << 1 radian. The Z-axis is directed from node i to node j, which are separated by a distance "L".

```
PretwistedBeamElementStiffnessGlobal[ L_, ϕi_, ϕj_, EIxx_, EIyy_, numer_ ] :=
        Module[
                { ∈, KeBar, sϕi, cϕi, Te, Ke },

                ∈ = ϕj - ϕi;
(*∈ = 0;*)
                KeBar = PretwistedBeamElementStiffnessLocal[ L, ∈, EIxx, EIyy, numer ];

                sϕi = Sin[ ϕi ];
                cϕi = Cos[ ϕi ];

                Te = { { -sϕi,    0,  cϕi,    0,    0,    0,    0,    0 },
                       {    0,  cϕi,    0,  sϕi,    0,    0,    0,    0 },
                       {    0,    0,    0,    0, -sϕi,    0,  cϕi,    0 },
                       {    0,    0,    0,    0,    0,  cϕi,    0,  sϕi },
                       {  cϕi,    0,  sϕi,    0,    0,    0,    0,    0 },
                       {    0, -sϕi,    0,  cϕi,    0,    0,    0,    0 },
                       {    0,    0,    0,    0,  cϕi,    0,  sϕi,    0 },
                       {    0,    0,    0,    0,    0, -sϕi,    0,  cϕi }  };
                If[ numer, Te = N[ Te ] ];

                Ke = Simplify[ Transpose[ Te ].KeBar.Te ];

                Return[ Ke ];
        ];
```

This Module assembles the master stiffness matrix of an initially straight (prior to load), pretwisted beam. The nodes are numbered sequentially from 1 to "numnod". The "numele" = numnod - 1 elements are placed sequentially between each pair of nodes (i.e., element 1 connects nodes 1 and 2, element 2 connects nodes 2 and 3, etc...). The displacement vector is assumed to be organized as follows:

```
{u} = Transpose[ {{ uX1     , θX1     , uY1     , θY1     ,
                   uX2     , θX2     , uY2     , θY2     ,
                   ...,
                   uXnumnod, θXnumnod, uYnumnod, θYnumnod  }} ]
```

The inputs are as follows:

"nodZ" - Z-location of each node
"nodϕ" - rotation of principle axes about the Z-axis of each node
"eleEIxx" - principle inertia about local x-axis of each element
"eleEIyy" - principle inertia about local y-axis of each element
"numer" - option whether to operate in floating point (True) or exact (false) arithmetic

```
PretwistedBeamMasterStiffness[ nodZ_, nodϕ_, eleEIxx_, eleEIyy_, numer_ ] :=
        Module[
                { numnod, numele, numDOF, K, i, EIxx, EIyy, L, ϕi, ϕj, Ke, row, col,
```

```
                rowM, colM },

            numnod = Length[ nodZ    ];
            numele = Length[ eleEIxx ];
            numDOF = 4*numnod;

            K = Table[ 0, {numDOF},{numDOF} ];
            For[ i = 1, i <= numele, i++,
                    EIxx = eleEIxx[[i  ]];
                    EIyy = eleEIyy[[i  ]];
                    L    = nodZ   [[i+1]] - nodZ[[i]];
                    ϕi   = nodϕ   [[i  ]];
                    ϕj   = nodϕ   [[i+1]];

                    Ke   = PretwistedBeamElementStiffnessGlobal[ L, ϕi, ϕj,
                                                         EIxx, EIyy, numer ];

                    For[ row = 1, row <= 8, row++,
                            For[ col = 1, col <= 8, col++,
                                    rowM = row + ( i - 1 )*4;
                                    colM = col + ( i - 1 )*4;
                                    K[[rowM,colM]] = K[[rowM,colM]] + Ke[[row,col]];
                            ];
                    ];
            ];

            Return[ Simplify[ K ] ];
    ];
```

This Module solves the problem of an initially straight (prior to load), pretwisted beam, which is cantilevered at node 1. The beam is loaded at the nodes as follows:

"nodFX" - force in the X-direction at each node
"nodMX" - moment about the X-axis at each node
"nodFY" - force in the Y-direction at each node
"nodMY" - moment about the Y-axis at each node

```
CantileveredPretwistedBeamSolution[ nodZ_, nodϕ_, eleEIxx_, eleEIyy_,
                    nodFX_, nodMX_, nodFY_, nodMY_, numer_ ] :=
    Module[
            { K, Kmod, numDOF, numnod, fmod, frecovered, i, j, u, uX, θX, uY, θY },

            K     = PretwistedBeamMasterStiffness[ nodZ, nodϕ,
                                        eleEIxx, eleEIyy, numer ];
            Kmod  = K;

            numDOF = Dimensions[ Kmod ][[1]];
            numnod = numDOF/4;
```

```
        fmod   = Table[ 0, {numDOF}, {1} ];
        For[ i = 1, i <= numnod, i++,
                fmod[[ 1 + ( i - 1 )*4 ]] = nodFX[[i]];
                fmod[[ 2 + ( i - 1 )*4 ]] = nodMX[[i]];
                fmod[[ 3 + ( i - 1 )*4 ]] = nodFY[[i]];
                fmod[[ 4 + ( i - 1 )*4 ]] = nodMY[[i]];
        ];

        For[ i = 1, i <= 4, i++,
                For[ j = 1, j <= numDOF, j++,
                        Kmod[[i,j]] = 0; (* all DOFs are fixed at zero for node 1 *)
                        Kmod[[j,i]] = 0;
                ];
                Kmod[[i,i]] = 1;
                fmod[[  i]] = 0;
        ];
        If[ numer, Kmod = N[ Kmod ] ];
        If[ numer, fmod = N[ fmod ] ];

        u = Simplify[ Inverse[ Kmod ].fmod ];

        frecovered = Simplify[ K.u ];

        uX = Table[ 0, {numnod} ];
        θX = Table[ 0, {numnod} ];
        uY = Table[ 0, {numnod} ];
        θY = Table[ 0, {numnod} ];
        For[ i = 1, i <= numnod, i++,
                uX[[i]] = u[[ 1 + ( i - 1 )*4 ]];
                θX[[i]] = u[[ 2 + ( i - 1 )*4 ]];
                uY[[i]] = u[[ 3 + ( i - 1 )*4 ]];
                θY[[i]] = u[[ 4 + ( i - 1 )*4 ]];
        ];

        Return[ { K, Kmod, fmod, u, uX, θX, uY, θY, frecovered } ];
];
```

Appendix C: *Mathematica* Driver Program for Bending of a Pretwisted, Cantilevered Beam with Small Overall Pretwist

This is the driver program for solution of a beam with uniform pretwist and homogenous elastic properties, which is loaded only at the tip. The beam is assembled using one element and the solution is performed analytically.

```
ClearAll[ L, EIxx, EIyy, Rootϕ, Tipϕ, TipFX, TipMX, TipFY, TipMY, numele, numer ];
ClearAll[ numnod, i, nodZ, nodϕ, eleEIxx, eleEIyy, nodFX, nodMX, nodFY, nodMY, K, Kmod,
fmod, u, uX, ΘX, uY, ΘY, frecovered ];
ClearAll[ P, ∈ ];

(* Inputs: *)

Rootϕ  = 0;
Tipϕ   = ∈;
TipFX  = 0;
TipMX  = 0;
TipFY  = -P;
TipMY  = 0;
numele = 1;
numer  = False;


(* Develop nodal and element properties: *)

numnod  = numele + 1;
nodZ    = Table[                    L*( i - 1 )/numele, {i,1,numnod} ];
nodϕ    = Table[ Rootϕ + ( Tipϕ - Rootϕ )*( i - 1 )/numele, {i,1,numnod} ];
eleEIxx = Table[ EIxx, {numele} ];
eleEIyy = Table[ EIyy, {numele} ];
nodFX   = Table[    0, {numnod} ];
nodMX   = Table[    0, {numnod} ];
nodFY   = Table[    0, {numnod} ];
nodMY   = Table[    0, {numnod} ];
nodFX[[numnod]] = TipFX;
nodMX[[numnod]] = TipMX;
nodFY[[numnod]] = TipFY;
nodMY[[numnod]] = TipMY;


(* Solve system: *)

{ K, Kmod, fmod, u, uX, ΘX, uY, ΘY, frecovered } = CantileveredPretwistedBeamSolution[
nodZ, nodϕ, eleEIxx, eleEIyy, nodFX, nodMX, nodFY, nodMY, numer ];
```

```
uX = Simplify[ uX /. { ∈^2 -> 0, ∈^3 -> 0, ∈^4 -> 0 }];
θX = Simplify[ θX /. { ∈^2 -> 0, ∈^3 -> 0, ∈^4 -> 0 }];
uY = Simplify[ uY /. { ∈^2 -> 0, ∈^3 -> 0, ∈^4 -> 0 }];
θY = Simplify[ θY /. { ∈^2 -> 0, ∈^3 -> 0, ∈^4 -> 0 }];

(* Print some results: *)

(*Print[ "K    = ", K    // MatrixForm ];
Print[ "Kmod = ", Kmod // MatrixForm ];
Print[ "fmod = ", fmod // MatrixForm ];
Print[ "u    = ", u    // MatrixForm ];*)
Print[ "uX   = ", uX   // MatrixForm ];
Print[ "θX   = ", θX   // MatrixForm ];
Print[ "uY   = ", uY   // MatrixForm ];
Print[ "θY   = ", θY   // MatrixForm ];
Print[ "frecovered = ", frecovered // MatrixForm ];
```

$$\mathrm{uX} \quad = \quad \begin{pmatrix} 0 \\ -\frac{(\mathrm{EIxx}-\mathrm{EIyy})\, L^3\, P\, \epsilon}{12\, \mathrm{EIxx}\, \mathrm{EIyy}} \end{pmatrix}$$

$$\theta\mathrm{X} \quad = \quad \begin{pmatrix} 0 \\ \frac{L^2\, P}{2\, \mathrm{EIxx}} \end{pmatrix}$$

$$\mathrm{uY} \quad = \quad \begin{pmatrix} 0 \\ -\frac{L^3\, P}{3\, \mathrm{EIxx}} \end{pmatrix}$$

$$\theta\mathrm{Y} \quad = \quad \begin{pmatrix} 0 \\ -\frac{(\mathrm{EIxx}-\mathrm{EIyy})\, L^2\, P\, \epsilon}{6\, \mathrm{EIxx}\, \mathrm{EIyy}} \end{pmatrix}$$

$$\mathrm{frecovered} \quad = \quad \begin{pmatrix} 0 \\ -L\, P \\ P \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \end{pmatrix}$$

Appendix D: *Mathematica* Driver Program for Bending of a Pretwisted, Cantilevered Beam with 90° Overall Pretwist

This is the driver program for solution of a beam with uniform pretwist and homogenous elastic properties, which is loaded only at the tip. The beam is assembled using "numele" equally-spaced elements.

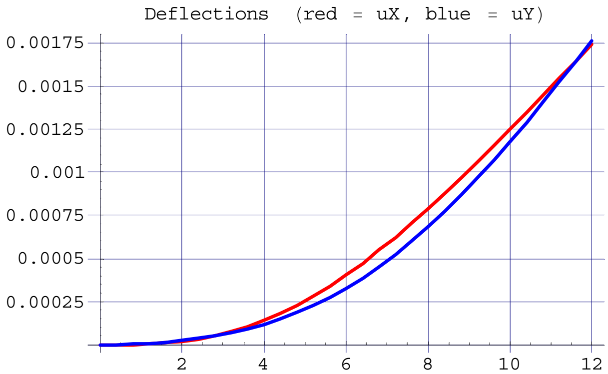
```
ClearAll[ L, h, b, Em, EIxx, EIyy, Rootϕ, Tipϕ, TipFX, TipMX, TipFY, TipMY, numele, numer
];
ClearAll[ numnod, i, nodZ, nodϕ, eleEIxx, eleEIyy, nodFX, nodMX, nodFY, nodMY, K, Kmod,
fmod, u, uX, θX, uY, θY, frecovered ];
ClearAll[ P, ∈ ];

(* Inputs: *)

L     = 12;
h     = 0.32;
b     = 1.1;
Em    = 29*10^6;
EIxx  = Em*h*(b^3)/12;
EIyy  = Em*b*(h^3)/12;

Rootϕ  =  0*Pi/180;
Tipϕ   = 90*Pi/180;
TipFX  = 0;
TipMX  = 0;
TipFY  = 1;
TipMY  = 0;
numele = 30;
numer  = True;


(* Develop nodal and element properties: *)

numnod  = numele + 1;
nodZ    = Table[                    L*( i - 1 )/numele, {i,1,numnod} ];
nodϕ    = Table[ Rootϕ + ( Tipϕ - Rootϕ )*( i - 1 )/numele, {i,1,numnod} ];
eleEIxx = Table[ EIxx, {numele} ];
eleEIyy = Table[ EIyy, {numele} ];
nodFX   = Table[    0, {numnod} ];
nodMX   = Table[    0, {numnod} ];
nodFY   = Table[    0, {numnod} ];
nodMY   = Table[    0, {numnod} ];
nodFX[[numnod]] = TipFX;
nodMX[[numnod]] = TipMX;
nodFY[[numnod]] = TipFY;
```

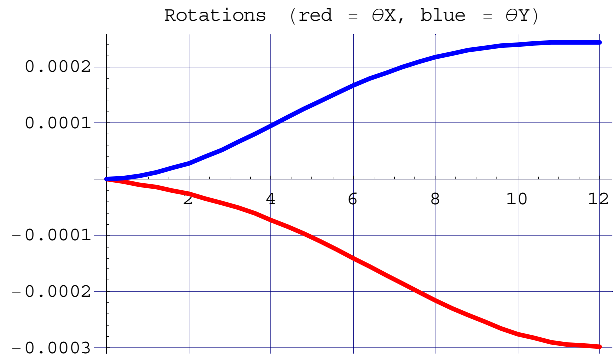
```
nodMY[[numnod]] = TipMY;


(* Solve system: *)

{ K, Kmod, fmod, u, uX, θX, uY, θY, frecovered } = CantileveredPretwistedBeamSolution[
nodZ, nodφ, eleEIxx, eleEIyy, nodFX, nodMX, nodFY, nodMY, numer ];


(* Print some results: *)

(*Print[ "K    = ", K    // MatrixForm ];
Print[ "Kmod = ", Kmod // MatrixForm ];
Print[ "fmod = ", fmod // MatrixForm ];
Print[ "u    = ", u    // MatrixForm ];*)
Print[ "uX   = ", uX   // MatrixForm ];
Print[ "θX   = ", θX   // MatrixForm ];
Print[ "uY   = ", uY   // MatrixForm ];
Print[ "θY   = ", θY   // MatrixForm ];
Print[ "frecovered = ", frecovered // MatrixForm ];


(* Plot some results: *)

plot1 = ListPlot[ Table[ {nodZ[[i]],uX[[i]]}, {i,1,numnod} ], PlotJoined -> True,
DisplayFunction -> Identity ];
plot2 = ListPlot[ Table[ {nodZ[[i]],uY[[i]]}, {i,1,numnod} ], PlotJoined -> True,
DisplayFunction -> Identity ];
Show[ Graphics[ AbsoluteThickness[2] ],
    Graphics[ RGBColor[1,0,0] ], plot1,
    Graphics[ RGBColor[0,0,1] ], plot2,
    DisplayFunction -> $DisplayFunction,
    TextStyle       -> { FontFamily -> "Courier New", FontSize -> 12 },
    PlotLabel       -> "Deflections (red = uX, blue = uY)",
    Axes            -> True,
    AxesOrigin      -> {0,0},
    GridLines       -> Automatic,
    PlotRange       -> All
];

plot1 = ListPlot[ Table[ {nodZ[[i]],θX[[i]]}, {i,1,numnod} ], PlotJoined -> True,
DisplayFunction -> Identity ];
plot2 = ListPlot[ Table[ {nodZ[[i]],θY[[i]]}, {i,1,numnod} ], PlotJoined -> True,
DisplayFunction -> Identity ];
Show[ Graphics[ AbsoluteThickness[3] ],
    Graphics[ RGBColor[1,0,0] ], plot1,
```

```
    Graphics[ RGBColor[0,0,1] ], plot2,
    DisplayFunction -> $DisplayFunction,
    TextStyle       -> { FontFamily -> "Courier New", FontSize -> 12 },
    PlotLabel       -> "Rotations (red = θX, blue = θY)",
    Axes            -> True,
    AxesOrigin      -> {0,0},
    GridLines       -> Automatic,
    PlotRange       -> All
];
```

$$\mathrm{uX} = \begin{pmatrix} 0. \\ 1.74714 \times 10^{-7} \\ 1.37225 \times 10^{-6} \\ 4.54052 \times 10^{-6} \\ 0.0000105364 \\ 0.0000201181 \\ 0.00003394 \\ 0.0000525482 \\ 0.000076379 \\ 0.000105759 \\ 0.000140905 \\ 0.000181932 \\ 0.000228852 \\ 0.000281585 \\ 0.000339965 \\ 0.000403749 \\ 0.000472628 \\ 0.000546237 \\ 0.000624167 \\ 0.000705976 \\ 0.0007912 \\ 0.000879369 \\ 0.000970015 \\ 0.00106269 \\ 0.00115695 \\ 0.00125242 \\ 0.00134874 \\ 0.00144562 \\ 0.00154282 \\ 0.00164017 \\ 0.00173756 \end{pmatrix} \quad \theta\mathrm{X} = \begin{pmatrix} 0. \\ -4.62741 \times 10^{-6} \\ -9.36255 \times 10^{-6} \\ -0.0000144396 \\ -0.0000200598 \\ -0.0000263897 \\ -0.0000335602 \\ -0.0000416657 \\ -0.0000507645 \\ -0.000060879 \\ -0.0000719971 \\ -0.0000840739 \\ -0.0000970334 \\ -0.000110772 \\ -0.000125159 \\ -0.000140045 \\ -0.000155259 \\ -0.000170616 \\ -0.000185923 \\ -0.000200976 \\ -0.000215573 \\ -0.000229511 \\ -0.000242592 \\ -0.000254629 \\ -0.000265446 \\ -0.000274886 \\ -0.000282809 \\ -0.000289097 \\ -0.000293655 \\ -0.000296416 \\ -0.000297338 \end{pmatrix} \quad \mathrm{uY} = \begin{pmatrix} 0. \\ 9.26486 \times 10^{-7} \\ 3.71687 \times 10^{-6} \\ 8.46189 \times 10^{-6} \\ 0.0000153395 \\ 0.0000246011 \\ 0.0000365581 \\ 0.0000515664 \\ 0.0000700131 \\ 0.0000923009 \\ 0.000118835 \\ 0.000150009 \\ 0.000186192 \\ 0.000227717 \\ 0.000274871 \\ 0.000327885 \\ 0.000386924 \\ 0.000452083 \\ 0.000523381 \\ 0.000600759 \\ 0.000684073 \\ 0.000773102 \\ 0.000867541 \\ 0.000967011 \\ 0.00107106 \\ 0.00117916 \\ 0.00129075 \\ 0.00140518 \\ 0.00152178 \\ 0.00163986 \\ 0.00175867 \end{pmatrix} \quad \theta\mathrm{Y} = \begin{pmatrix} 0. \\ 1.30894 \times 10^{-6} \\ 5.09357 \times 10^{-6} \\ 0.000011137 \\ 0.0000192015 \\ 0.0000290329 \\ 0.0000403652 \\ 0.0000529254 \\ 0.0000664378 \\ 0.0000806288 \\ 0.0000952311 \\ 0.000109988 \\ 0.000124658 \\ 0.000139015 \\ 0.000152858 \\ 0.000166006 \\ 0.000178305 \\ 0.000189632 \\ 0.000199889 \\ 0.00020901 \\ 0.000216959 \\ 0.000223729 \\ 0.000229343 \\ 0.000233851 \\ 0.00023733 \\ 0.000239879 \\ 0.00024162 \\ 0.000242691 \\ 0.000243247 \\ 0.000243453 \\ 0.000243482 \end{pmatrix}$$

$$\mathrm{frecovered} = \begin{pmatrix} -1.32843 \times 10^{-10} \\ 12. \\ -1. \\ -1.50477 \times 10^{-9} \\ -1.1571 \times 10^{-13} \\ -2.10415 \times 10^{-13} \\ -7.25573 \times 10^{-12} \\ -7.91554 \times 10^{-13} \\ -6.22528 \times 10^{-12} \\ -1.27608 \times 10^{-11} \\ 5.70796 \times 10^{-11} \\ -1.35408 \times 10^{-12} \\ 8.12626 \times 10^{-12} \\ 2.31329 \times 10^{-12} \\ -1.15088 \times 10^{-10} \\ -1.52122 \times 10^{-12} \\ 4.87405 \times 10^{-11} \\ 3.43181 \times 10^{-11} \\ -6.15335 \times 10^{-11} \\ 9.43648 \times 10^{-12} \\ -1.26562 \times 10^{-10} \\ -2.21412 \times 10^{-11} \\ 3.50418 \times 10^{-10} \\ -5.03199 \times 10^{-12} \\ 3.23607 \times 10^{-10} \\ 8.2782 \times 10^{-11} \\ -8.5749 \times 10^{-10} \\ 3.32271 \times 10^{-11} \\ -6.24356 \times 10^{-10} \\ -4.90629 \times 10^{-11} \\ 1.49654 \times 10^{-9} \\ -2.40741 \times 10^{-11} \\ 6.95758 \times 10^{-10} \\ -3.93908 \times 10^{-11} \\ -1.54427 \times 10^{-9} \\ -1.54854 \times 10^{-11} \\ -2.19201 \times 10^{-10} \\ 1.799 \times 10^{-10} \\ 4.49727 \times 10^{-10} \\ 8.36173 \times 10^{-11} \\ -4.61976 \times 10^{-10} \\ -8.69761 \times 10^{-11} \\ 8.86329 \times 10^{-10} \\ -5.06507 \times 10^{-11} \\ 4.56801 \times 10^{-10} \\ -9.06524 \times 10^{-11} \\ -8.6718 \times 10^{-10} \\ -5.51134 \times 10^{-11} \\ -2.89009 \times 10^{-11} \\ 4.92012 \times 10^{-11} \\ 1.62788 \times 10^{-10} \\ 3.25402 \times 10^{-11} \\ 1.76714 \times 10^{-11} \\ 3.33108 \times 10^{-11} \\ -1.15633 \times 10^{-10} \\ 2.62961 \times 10^{-11} \\ 1.50646 \times 10^{-10} \\ 6.71579 \times 10^{-11} \\ -4.84564 \times 10^{-11} \\ 6.23296 \times 10^{-11} \\ -7.0317 \times 10^{-10} \\ -5.07849 \times 10^{-11} \\ 6.50206 \times 10^{-10} \\ -4.48883 \times 10^{-11} \\ -1.16989 \times 10^{-10} \\ -1.83227 \times 10^{-10} \\ -1.72691 \times 10^{-11} \\ -2.07992 \times 10^{-10} \\ 1.22082 \times 10^{-9} \\ 1.51969 \times 10^{-11} \\ -1.0201 \times 10^{-9} \\ 1.00215 \times 10^{-11} \\ -1.20108 \times 10^{-9} \\ 1.63716 \times 10^{-11} \\ 1.01221 \times 10^{-9} \\ 1.44932 \times 10^{-11} \\ 1.22914 \times 10^{-9} \\ 2.37791 \times 10^{-11} \\ -1.00725 \times 10^{-9} \\ 2.84077 \times 10^{-11} \\ -1.05485 \times 10^{-9} \\ 5.52342 \times 10^{-11} \\ 8.11389 \times 10^{-10} \\ 6.17011 \times 10^{-11} \\ 2.38952 \times 10^{-10} \\ -7.35058 \times 10^{-11} \\ -1.64359 \times 10^{-10} \\ -9.40888 \times 10^{-11} \\ 1.24014 \times 10^{-10} \\ -1.96005 \times 10^{-11} \\ -6.69085 \times 10^{-11} \\ -2.5661 \times 10^{-11} \\ 2.68029 \times 10^{-10} \\ 1.46403 \times 10^{-11} \\ -1.48578 \times 10^{-10} \\ 2.44558 \times 10^{-11} \\ -1.5099 \times 10^{-10} \\ 3.57482 \times 10^{-11} \\ 8.68587 \times 10^{-11} \\ 4.80543 \times 10^{-11} \\ 9.20686 \times 10^{-12} \\ 1.13465 \times 10^{-11} \\ 5.07439 \times 10^{-12} \\ 2.59943 \times 10^{-11} \\ -8.51905 \times 10^{-11} \\ -9.76594 \times 10^{-13} \\ 3.79318 \times 10^{-11} \\ 7.28206 \times 10^{-12} \\ -1.41505 \times 10^{-11} \\ -3.18137 \times 10^{-12} \\ 4.12448 \times 10^{-13} \\ -2.98195 \times 10^{-11} \\ 8.14513 \times 10^{-11} \\ -3.12819 \times 10^{-13} \\ -1.20182 \times 10^{-11} \\ -2.38499 \times 10^{-11} \\ 1.83249 \times 10^{-11} \\ -5.91145 \times 10^{-13} \\ 2.52964 \times 10^{-13} \\ -3.62077 \times 10^{-12} \\ -1.40901 \times 10^{-11} \\ 7.90687 \times 10^{-14} \\ 1. \\ -1.49925 \times 10^{-12} \end{pmatrix}$$


Deflections (red = uX, blue = uY)
0.00175
0.0015
0.00125
0.001
0.00075
0.0005
0.00025
2
4
6
8
10
12


Rotations (red = θX, blue = θY)
0.0002
0.0001
-0.0001
-0.0002
-0.0003
2
4
6
8
10
12